JEDNA NOVA FORMULACIJA PRINCIPA INDUKCIJE

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1. Ovde ćemo dati neka uopštenja u vezi sa pojmom „induktivnog sistema", uvedenim u jednom ranijem našem radu.[1])

Najpre ćemo dati neka objašnjenja. Pojmovima klase i množine pridajemo smisao koji imaju u GÖDEL-ovom sistemu aksioma teorije množina.[2]) Sledeće oznake ćemo upotrebljavati: Λ — prazna klasa, V — univerzalna klasa; $A \setminus B$ — klasa svih elemenata klase A koji ne pripadaju klasi B; $P(A)$ — partitivna klasa, tj. klasa svih potklasa klase A; $S_A = S \cap P(A)$, gde su S i A proizvoljne klase. Termini „sistem" i „klasa" su sinonimi.

Ako je p neki iskaz (propozicija), $\sim p$ je njegova negacija.

U uređenom paru (a, b) a je leva (ili prva), a b desna (ili druga) komponenta para.

Navodimo još neke definicije.

Definicija 1. 1. Neka su A i B proizvoljne klase. *Kombinovani produkt* $A \times B$ datih klasa je sistem svih uređenih parova (x, y), $x \in A$, $y \in B$. $A \times A = A^2$.

Definicija 1. 2. Svaka klasa $\varphi \subseteq V^2$ naziva se *binarnom relacijom*. *Levi (desni)*[3]) *domen* $D\varphi$ $(W\varphi)$ relacije φ je klasa svih levih (desnih) komponenata njenih elemenata. Ako je A neka klasa onda je $D_A\varphi$ $(W_A\varphi)$ klasa levih (desnih) komponenata svih onih elemenata od φ čije desne (leve) komponente pripadaju klasi A. Izrazi $(x, y) \in \varphi$ i $x \varphi y$ su ekvivalentni.

Ako je ρ binarna relacija izraz $a, b, \cdots \rho\, x$ $(x \rho\, a, b \cdots)$ je ekvivalentan sistemu relacija $a \rho x$, $b \rho x, \cdots (x \rho a, x \rho b, \cdots)$.

2. Opšta šema principa indukcije glasi:

Propozicija 2. 1. Neka su M, S, S_1 i N proizvoljne klase a φ binarna relacija. Iz uslova:

1. $S_{M \cap N} \cap S_1 \neq \Lambda, \{\Lambda\}$;

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[1]) Popadić, M., O induktivnim sistemima. Годишен зборник, Филозофски факултет на Универзитетот — Скопје. Природно-математички оддел. T. 7, № 1, 1954.

O indukciji vidi i članak Đ. Kurepe: O principima indukcije (Zbornik radova matematičkog instituta Srpske akademije nauka, br. 1, 1951, str. 109—118).

[2]) Gödel, K., The Consistency of the Continuum Hypothesis. Annals of Mathematics Studies, № 3. Princeton University Press, Princeton N. J., 1940.

[3]) Čitanjem termina u zagradama mesto onih ispred zagrada, dobijaju se dualni stavovi.

2. postoji relacija $\psi\in P(\varphi)$ sa levim domenom $D\psi=S_M\setminus\{\Lambda, M\}$ za koju je $W_{S_{M\cap N}\setminus\{\Lambda,M\}}\psi\subseteq P(M\cap N)$ — sleduje $M\subseteq N$.

U vezi sa ovim iskazom imamo sledeću definiciju:

Definicija 2. 1. Uređena četvorka (M, S, S_1, φ), gde su M, S i S_1 proizvoljne klase a φ binarna relacija naziva se *induktivnom* ako je propozicija 2. 1 istinita. S je *induktivni sistem*, S_1 — *baza indukcije*, φ *induktor*. Ovi nazivi imaju smisla samo u okviru induktivne četvorke.

Ako je $S, S_1\subseteq P(M)$ i $\varphi\subseteq P(M)\times P(M)$ onda se propozicija 2. 1 i definicija 2. 1 svode na propoziciju 2. 2. 1 i definiciju 2. 2. 1 u rezimeu ranije navedenog našeg članka.[4]) Definiciji potencijalnog sistema u istom radu (D 8. 6. 2 i u rezimeu D 3. 2. 2) odgovara sledeća definicija:

Definicija 2. 2. Neka su M, S, S_1 proizvoljne klase a φ binarna relacija. Uređena četvorka (M, S, S_1, φ) naziva se *potencijalnim* ako je za svako $\psi\in P(\varphi)$ sa $D\psi=S_M\setminus\{\Lambda, M\}$ zadovoljena relacija $\sim(W_{S_E\setminus\{\Lambda\}}\psi\subseteq P(E))$ samo ako je $E\subset M\cap\bigcup_{X\in W\psi} X$ i $S_E\cap S_1\neq\Lambda, \{\Lambda\}$. Klasa S naziva se potencijalnim sistemom, a S_1 i φ bazom indukcije, odnosno induktorom.

3. Da bi formulisali osnovni stav u vezi sa induktivnim četvorkama, potrebna nam je sledeća definicija:

Definicija 3. 1. Neka je A proizvoljna klasa a S izvestan sistem klasa. Ako je $A\subseteq\bigcup_{X\in S} X$, kaže se da je S prekrivač od A, ili da S prekriva A.

Osnovni stav glasi:

Teorema 3. 1. *Da bi uređena četvorka (M, S, S_1, φ), gde su M, S i S_1 proizvoljne klase a φ binarna relacija, bila induktivna nužno je i dovoljno da je potencijalna i da, za svako $\psi\in P(\varphi)$ sa $D\psi=S_M\setminus\{\Lambda, M\}$, $W\psi$ prekriva M.*

Mada je dokaz ovoga stava formalno isti kao i u citiranom radu, radi potpunosti mi ćemo ga izvesti.

Dokaz. Uslov je nužan. Pretpostavimo najpre da je data četvorka induktivna, ali da postoji bar jedna ralacija $\psi\in P(\varphi)$ sa levim domenom $D\psi=S_M\setminus\{\Lambda, M\}$, takva da sistem $W\psi$ ne prekriva M. Dakle tada je $\sim(M\subseteq\bigcup_{X\in W\psi} X)$. Stavljajući $M\cap\bigcup_{X\in W\psi} X=N$ imamo

$$N\subset M. \tag{3. 1}$$

Ne protivrečeći dosadašnjim postavkama, možemo pretpostaviti da je $S_N\cap S_1\neq\Lambda, \{\Lambda\}$ i $W_{S_N\setminus\{\Lambda\}}\psi\subseteq P(N)$. Ove relacije, s obzirom na (3. 1), pretstavljaju upravo uslove 1 i 2 propozicije

[4]) I u srpskom tekstu postoje odgovarajući iskazi (Pr 8. 6. 1 i D 8. 6. 1), samo što je tu mesto pojma binarne relacije upotrebljen pojam transformacije.

2. 1. Međutim zbog induktivnosti date četvorke sledovalo bi da je $M \subseteq N$, što protivreči relaciji (3. 1).

Pretpostavimo opet da je data četvorka induktivna, ali da nije potencijalna. Dakle postoji relacija $\psi \in P(\varphi)$ sa levim domenom $D\psi = S_M \setminus \{\Lambda, M\}$ i jedna klasa $N \subset M \cap \bigcup_{X \in W\psi} X$, odnosno

$$(3.\ 2) \qquad N \subset M$$

takva da je ipak, iako je

$$(3.\ 3) \qquad S_N \cap S_1 \neq \Lambda,\ \{\Lambda\},$$

zadovoljena relacija

$$(3.\ 4) \qquad W_{S_N \setminus \{\Lambda\}} \psi \subseteq P(N).$$

Iz relacija (3. 2) i (3. 3) jasno je da je za klase M i N zadovoljen uslov 1 propozicije 2. 1. Pošto je dalje zbog (3. 2) $P(M \cap N) = P(N)$, sleduje iz (3. 4) da je i uslov 2 iste propozicije takođe zadovoljen. Kako je po pretpostavci data četvorka induktivna, bilo bi tada $M \subseteq N$, što protivreči relaciji (3. 2). Ovim je dokazana nužnost uslova navedenih u stavu.

Uslov je dovoljan. Neka je data četvorka potencijalna i pretpostavimo da za svako $\psi \in P(\varphi)$ sa $D\psi = S_M \setminus \{\Lambda, M\}$ sistem $W\psi$ prekriva M, tj. da je

$$(3.\ 5) \qquad M \subseteq \bigcup_{X \in W\psi} X$$

Neka su takođe i uslovi 1 i 2 propozicije 2. 1 zadovoljeni, ali neka je ipak

$$(3.\ 6) \qquad \sim (M \subseteq N),$$

tj. data četvorka nije induktivna. Iz (3. 6) imamo

$$(3.\ 7) \qquad M \cap N = E \subset M,\ E \subseteq N$$

i, saglasno uslovu 1. 2 pomenute propozicije, $S_{M \cap N} \cap S_1 \neq \Lambda,\ \{\Lambda\}$, odnosno zbog (3. 7)

$$(3.\ 8) \qquad S_E \cap S_1 \neq \Lambda,\ \{\Lambda\}.$$

Prema uslovu 2 iste propozicije postoji relacija $\psi \in P(\varphi)$ sa levim domenom $D\psi = S_M \setminus \{\Lambda, M\}$ za koju je $W_{S_{M \cap N} \setminus \{\Lambda, M\}} \psi \subseteq P(M \cap N)$ ili, zbog (3. 7),

$$(3.\ 9) \qquad W_{S_E \setminus \{\Lambda\}} \psi \subseteq P(E).$$

Međutim kako je, na osnovu (3. 5) i (3. 7), $E \subset M \cap \bigcup_{X \in W\psi} X$ i pošto je zadovoljena relacija (3. 8), zbog potencijalnosti date četvorke sleduje $\sim (W_{S_E \setminus \{\Lambda\}} \psi \subseteq P(E))$, što protivreči relaciji (3. 9). Ovim je teorema u potpunosti dokazana.

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A NEW FORMULATION OF THE PRINCIPLE OF INDUCTION

(Summary)

1. Here we present generalisations in connection with the notion of the „inductive system“.[1]

The meaning of the notions of a class and of a set will be taken as in GÖDEL axiom system understood.[2]

We shall use the following symbols: Λ — empty class, V — universal class; $A \setminus B$ — the class of all elements of A not in B; $P(A)$ — the class of all subclass of A; $S_A = S \cap P(A)$, where S and A are any classes. The terms „system“ and „class“ are considered as synonymous. $\sim p$ is the negation of the proposition p.

In a couple (a, b) we call a and b the left and right component, respectively.

Definition 1. 1. By the Cartesian product $A \times B$ of the class A and B is meant the class of all couples (x, y), $x \in A$, $y \in B$. $A \times A = A^2$.

Definition 1. 2. A binary relation is every class $\varphi \subseteq V^2$. By the left (right)[3] domain $D\varphi$ $(W\varphi)$ of a binary relation φ we mean the class of left (right) components of all elements of φ. If A is any class, then $D_A \varphi$ $(W_A \varphi)$ represents the class of the left (right) components of all elements of φ, whose right (left) components belong to A.

ρ being a binary relation, the expression $a, b, \cdots \rho\, x$ $(x \rho\, a, b, \cdots)$ is equivalent to the system of expressions $a \rho x, b \rho x, \cdots (x \rho a, x \rho b, \cdots)$.

2. The general scheme of the princip'e of induction is as follows:

Proposition 2. 1. Let M, S, S_1 and N be any classes and φ a binary relation. From the conditions:

1. $S_{M \cap N} \cap S_1 \neq \Lambda, \{\Lambda\}$;

2. there exists a relation $\psi \in P(\varphi)$ with the left domain $D\psi = S_M \setminus \{\Lambda, M\}$, satisfying $W_{S_{M \cap N} \setminus \{\Lambda, M\}} \psi \subseteq P(M \cap N)$

— it follows $M \subseteq N$.

In connection with this proposition we have the following definition:

Definition 2. 1. A quadruple (M, S, S_1, φ), where M, S and S_1 are any classes and φ a binary relation, is inductive if the proposition 2. 1 is true. Then S is called the inductive system, S_1 — the basis of induction, φ — the inductor.

If $S, S_1 \subseteq P(M)$ and $\varphi \subseteq P(M) \times P(M)$, then we obtain the proposition 2. 2. 1 and the def nition 2. 2. 1 in the summary of the quoted paper.

Definition 2. 2.[4] Let M, S and S_1 be any classes and φ a binary relation. A quadruple (M, S, S_1, φ) is potential if the relation $\sim (W_{S_E \setminus \{\Lambda\}} \psi \subseteq P(E))$ is satisfied for every $\psi \in P(\varphi)$ with $D\psi = S_M \setminus \{\Lambda, M\}$, provided $E \subset M \cup \bigcap_{X \in W\psi} X$ and $S_E \cap S_1 \neq \Lambda, \{\Lambda\}$. The system S is potential, S_1 is the basis of induction, φ — the inductor.

3. In order to formulate the fundamental theorem, we cite the definition:

Definition 3. 1. A systems of classes covers a class A if $A \subseteq \bigcup_{X \in S} X$. S is a covering of A.

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[1]) Popadić, M., On the inductive systems. Faculté de philosophie de l'Université de Skopje. Section des sciences naturelles. T. 7, № 1, 1954.

Also see the paper G. Kurepa's: O principima indukcije (Zb. radova matematičkog instituta Srpske akademije nauka, br. 1 1951, str. 109—118).

[2]) See the footnote [2]) on the page 3.

[3]) By reading terms in brackets instead those before brackets, one obtains a dual statement.

[4]) In the mentioned summary this definition coresponds to D 3. 2. 2.

The basic theorem is as follows:

Theorem 3. 1. In order that a quadruple (M, S, S_1, φ) where M, S and S_1 are any classes and φ a binary relation, is inductive it is necessary and sufficient that it is potential and that the system $W\psi$ covers M for every $\psi \in P(\varphi)$ with $D\psi = S_M \setminus \{\Lambda, M\}$.

Proof. The condition is necessary. At first, suppose that the quadruple (M, S, S_1, φ) is inductive, but yet there is $\psi \in P(\varphi)$ with $D\psi = S_M \setminus \{\Lambda, M\}$ so that the system $W\psi$ do not cover M. Then $\sim (M \subseteq \bigcup_{X \in W\psi} X)$. If $M \cap \bigcup_{X \in W\psi} X = N$, we have

$$(3.1) \qquad N \subset M.$$

We may suppose that $S_N \cap S_1 \neq \Lambda, \{\Lambda\}$ and $W_{S_N \setminus \{\Lambda\}} \psi \subseteq P(N)$ which do not contradict our previous assumptions. These relations, according to (3. 1), represent just the conditions 1 and 2 of the proposition 2. 1. However, for the given quadruple is inductive, it follows that $M \subseteq N$, contrary to (3. 1).

Let, now, the given quadruple be inductive, but let us assume that it is not potential. Thus there is a binary relation $\psi \in P(\varphi)$ with the left domain $D\psi = S_M \setminus \{\Lambda, M\}$ and a class $N \subset M \cap \bigcup_{X \in W\psi} X$, or

$$(3.2) \qquad N \subset M,$$

such that nevertheless, though it is

$$(3.3) \qquad S_N \cap S_1 \neq \Lambda, \{\Lambda\},$$

the following relation holds

$$(3.4) \qquad W_{S_N} \setminus \{\Lambda\} \psi \subseteq P(N)$$

From the relations (3. 2) and (3. 3) it is obviously that the condition 1 of the proposition 2. 1 is satisfied. Since, because of (3. 2), $P(M \cap N) = P(N)$, it follows from (3. 4) that the condition 2 of the same proposition is satisfied too. Then, for the given quadruple is inductive, we have $M \subseteq N$. This contradicts (3. 2). The necessity of the condition is proved.

The condition is sufficient. Let the given quadruple be potential and let us assume that, for each $\psi \in P(\varphi)$ with $D\psi = S_M \setminus \{\Lambda, M\}$, the system $W\psi$ covers M, i. e.,

$$(3.5) \qquad M \subseteq \bigcup_{X \in W\psi} X.$$

Let the conditions 1 and 2 of the proposition 2. 1 be satisfied too, but let us assume that

$$(3.6) \qquad \sim (M \subseteq N)$$

i. e., the given quadruple is not inductive. From (3. 6) we have

$$(3.7) \qquad M \cap N = E \subset M, \ E \subseteq N$$

and, according to the condition 1 of the mentioned proposition, $S_{M \cap N} \cap S_1 \neq \Lambda, \{\Lambda\}$, or, because of (3. 7),

$$(3.8) \qquad S_E \cap S_1 \neq \Lambda, \{\Lambda\}.$$

In consequence of the condition 2 of the same proposition there is a binary relation $\psi \in P(\varphi)$ with $D\psi = S_M \setminus \{\Lambda, M\}$, satisfying $W_{S_{M \cap N} \setminus \{\Lambda\}} \psi \subseteq P(M \cap N)$ or, because of (3. 7),

$$(3.9) \qquad W_{S_E \setminus \{\Lambda\}} \psi \subseteq P(E).$$

In consequence of (3. 5) and (3. 7) we have $E \subset M \cap \bigcup_{X \in W\psi} X$. Hence and from (3. 8), since the given quadruple is potential, it follows $\sim (W_{S_E \setminus \{\Lambda\}} \psi \subset P(E))$, which contradicts (3. 9). Thus the theorem is true.