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MODIFIED BASKAKOV-KANTOROVICH OPERATORS PROVIDING A BETTER ERROR ESTIMATION

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Abstract. We introduce a kind of Baskakov-Kantorovich operators, which preserve the test functions 1 and x^2. This type of modification enables better error estimation on the interval $[\frac{\sqrt{3}}{3}, +\infty)$ than the classic ones. Finally, a Voronovskaya-type theorem for these operators is also obtained.

1. Introduction

King-type approximation operators[1-7] preserving the test functions 1 and x^2, and have better approximation properties than the classical ones. Motivated by this, we introduce a kind of Baskakov- Kantorovich operators, which preserve the test functions 1 and x^2 and have better error estimation on the interval $[\frac{\sqrt{3}}{3}, +\infty)$ than the classical Baskakov-Kantorovich operators.

Then, the classical Baskakov-Kantorovich operators are defined by

$$V_n^*(f,x) = n\sum_{k=0}^{\infty} v_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{1.1}$$

where $v_{n,k}(x) = \binom{n+k-1}{k}x^k(1+x)^{-n-k}$, $f \in C_\beta[0,+\infty) := \{f \in C[0,+\infty) : |f(t)| \leq M(1+t)^\beta \text{ for some } M > 0, \beta > 0\}$.

Let $f \in C_\beta[0,+\infty)$, $u_n(x) = \frac{-1+\sqrt{n(n+1)x^2+\frac{2}{3}-\frac{1}{3n}}}{n+1}$, $x \geq \frac{\sqrt{3}}{3}$ and $n \in N$, then we get the following modified positive linear operators:

$$V_n^{**}(f,x) = n\sum_{k=0}^{\infty} v_{n,k}(u_n(x))\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt \tag{1.2}$$

We will give the moments and convergence theorem of our operators, which preserving the test functions 1 and x^2.

The main result of this paper is:

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Theorem 1. *Let $f \in C_B[0,+\infty)$, the space of all bounded functions on $[0,+\infty)$, for $x \geq \frac{\sqrt{3}}{3}, n \in N$, we have*

$$|V_n^{**}(f,x) - f(x)| \leq 2\omega\left(f, \sqrt{\frac{\varphi^2(x)}{n} - \frac{1}{2n^2}}\right),$$

where $\varphi^2(x) = x(1+x)$ for V_n^{**}, the modulus of continuity of f denoted by $\omega(f,\delta_x)$ for $\delta_x > 0$, is defined to be

$$\omega(f,\delta_x) = \sup_{|t-x|\leq\delta_x, t\in[0,+\infty)} |f(t) - f(x)|.$$

Throughout this paper, M denotes a positive constant independent of n and x and not necessarily the same at each occurrence.

2. The convergence theorem of the modified operators

By calculation, we can obtain the following result.

Lemma 1. *For each $x \geq \frac{\sqrt{3}}{3}$, we have*
(1). $V_n^{**}(1,x) = 1$;
(2). $V_n^{**}(t,x) = \frac{\sqrt{n(n+1)x^2+2/3-\frac{1}{3n}}}{n+1} - \frac{n-1}{2n(n+1)}$;
(3). $V_n^{**}(t^2,x) = x^2$.

By Lemma 1, it is clear that the operators V_n^{**} given by (1.2) preserve the test functions 1 and x^2. Then from Lemma 1, one can get the following results for moments.

Lemma 2. *For each $x \geq \frac{\sqrt{3}}{3}$, we have*
(1). $V_n^{**}(t-x,x) = \frac{\sqrt{n(n+1)x^2+2/3-\frac{1}{3n}}}{n+1} - \frac{n-1}{2n(n+1)} - x$;
(4). $V_n^{**}((t-x)^2,x) = 2x^2 - \frac{x}{n} - 2xu_n(x)$;
(3). $V_n^{**}((t-x)^2,x) \leq \frac{\varphi^2(x)}{n} - \frac{1}{2n^2}$.

From Lemma 1, 2 and with the Korovkin-type property, we have the following convergence theorem.

Theorem 2. *Let $f \in C_\beta[0,+\infty)$, $x \geq \frac{\sqrt{3}}{3}$, we have $\lim_{n\to\infty} V_n^{**}(f,x) = f(x)$.*

3. Better error estimation

Theorem 3. *Let $f \in C_B[0,+\infty)$, $x \geq \frac{\sqrt{3}}{3}, n \in N$, we have*

$$|V_n^{**}(f,x) - f(x)| \leq 2\omega\left(f, \delta_{n,x}\right),$$

where $\delta_{n,x} = \sqrt{\frac{\varphi^2(x)}{n} - \frac{1}{2n^2}}$.

Proof. Let $f \in C_B[0,+\infty)$ and $x \geq 0$, using linearity and monotonicity of the operators V_n^{**}, for every $\delta > 0, n \in N$, we get

$$|V_n^{**}(f,x) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{V_n^{**}((t-x)^2,x)}\right).$$

Applying Lemma 2 and choosing $\delta = \delta_{n,x}$, the proof is complete. □

Remark 3.1. (1).[8] For the Baskakov-Kantorovich operators given by (1.1), we may write that, for $f \in C_B[0,+\infty)$ and $x \geq 0$, $n \in N$,

$$|V_n^{*}(f,x) - f(x)| \leq 2\omega\left(f, \alpha_{n,x}\right), \tag{3.1}$$

where $\alpha_{n,x} = \sqrt{\frac{x(1+x)}{n} + \frac{1}{3n^2}}$.

(2).We can see that the error estimation in Theorem 3 is better than that of (3.1) provided $f \in C_B[0,+\infty), x \geq \frac{\sqrt{3}}{3}$.

Indeed, it is clear that

$$\frac{x(1+x)}{n} - \frac{1}{2n^2} < \frac{x(1+x)}{n} + \frac{1}{3n^2}, \tag{3.2}$$

which guarantees that $\delta_{n,x} < \alpha_{n,x}$ for $x \geq \frac{\sqrt{3}}{3}$.

We say that a bounded function $f \in C[0,+\infty)$ belongs to $Lip_M(\alpha)$ if the inequality $|f(t) - f(x)| \leq M|t-x|^{\alpha}$ holds for all $t \in [0,+\infty)$.

Theorem 4. *For every $f \in Lip_M(\alpha)$, $x \geq \frac{\sqrt{3}}{3}$ and $n \in N$, we have*

$$|V_n^{**}(f,x) - f(x)| \leq M\left\{\frac{\varphi^2(x)}{n} - \frac{1}{2n^2}\right\}^{\frac{\alpha}{2}}.$$

Proof. Since $f \in Lip_M(\alpha)$, $x \geq 0$, using the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned}&|V_n^{**}(f,x) - f(x)| \leq V_n^{**}(|f(t) - f(x)|, x) \leq MV_n^{**}(|t-x|^{\alpha}, x)\\ &\leq M\left(V_n^{**}(|t-x|^2,x)\right)^{\frac{\alpha}{2}} \leq M\left\{\frac{\varphi^2(x)}{n} - \frac{1}{2n^2}\right\}^{\frac{\alpha}{2}}.\end{aligned}$$

□

Remark 3.2. The classical Baskakov-Kantorovich operators given by (1.1) satisfy

$$|V_n^{*}(f,x) - f(x)| \leq M\left\{\frac{x(1+x)}{n} + \frac{1}{3n^2}\right\}^{\frac{\alpha}{2}}, \tag{3.3}$$

respectively for $f \in Lip_M(\alpha)$, $x \geq \frac{\sqrt{3}}{3}$ and $n \in N$.

It follows from (3.2) that the rate of convergence of the operators V_n^{**} for the Lipschitz class functions is better than the error estimation given by (3.3) whenever $x \geq \frac{\sqrt{3}}{3}$.

4. A Voronovskaya-type Theorem

Along the same lines of the proof of Theorem 4.2 in [1], we have a Voronovskaya-type theorem for the operators V_n^{**} given by (1.2).

Theorem 5. *For every $f \in C_\beta[0,+\infty)$ such that $f^{'}, f^{''} \in C_\beta[0,+\infty)$, we have*

$$\lim_{n\to\infty} n\left\{V_n^{**}(f,x) - f(x)\right\} = -\frac{1}{2}f^{'}(x) + \frac{\varphi^2(x)}{2}f^{''}(x)$$

uniformly with respect to $x \in [\frac{\sqrt{3}}{3}, b](b > \frac{\sqrt{3}}{3})$.

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