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FUNCTIONS PRESERVING CONNECTEDNESS OR PATH-CONNECTEDNESS

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Abstract

We prove some theorems and give examples concerning functions preserving connectedness and functions preserving path connectedness.

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Only a few authors, mentioned that the converse is also true, mainly independently one from another.

For real functions $f\colon \mathbf{R} \to \mathbf{R}$ the first result of this type is given by Rowe in 1926. In 1965 Whyburn proved this result for 1-countable and locally connected space X, and Y Hausdorff. A generalization of this result is given by McMillan in 1970 and by Janos Gerlits, Istvan Juhasz, Lajos Soukup, and Zoltan Szentmiklossy.

In the paper a map means a continuous function.

Definition. The function $f\colon X \to Y$ is *connected,* if for any connected set S in X, $f(S)$ is connected.

The restriction of a connected function is a connected function. The composition of connected functions is a connected function.

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It is known that for a continuous map $f: X \to Y$, an image of compact set is compact and an image of connected set is connected.

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The restriction of a connected function is a connected function. The composition of connected functions is a connected function.

Example 1. A derivative of a real function is a connected function. For example, the real differentiable function

$$F(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

has a derivative

$$F'(x) = f(x) = \begin{cases} \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

which has a discontinuity at 0.

That the derivative is connected follows from theorem due to Darboux: If $F'(x) = f(x)$ for all $x \in (a, b)$, then for a pair $u, \nu \in (a, b)$ such that $u < \nu$ and $f(u) < d < f(\nu)$ (or $f(\nu) < d < f(u)$) there is $w \in (u, \nu)$ such that $f(w) = d$.

As a corollary we obtain: If $F'(x) = f(x)$ for all $x \in (a, b)$, then for any interval $[u, \nu] \subseteq (a, b)$, the image is $f([u, \nu])$ is an interval.

Since the connected subsets of real line are intervals, we obtain that $F'(x) = f(x)$ is a connected function.

Theorem 1. If X, Y are Hausdorff spaces and $f: X \to Y$ is a connected function, then $f(\overline{C}) \subseteq \overline{f(C)}$ for any connected subset C of X.

Proof. [5], Theorem 3.3.

Example 2. The Dirichlet function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbf{Q} \\ 1, & x \in \mathbf{R} \setminus \mathbf{Q}, \end{cases} \qquad f: \mathbf{R} \to \mathbf{R}$$

is not a connected function $\left(f\big((0,1)\big) = \{0,1\}\right)$. In spite of this, $f(\overline{C}) \subseteq \overline{f(C)}$, for any connected subset C of $\mathbf{R}$. The connected subsets of $\mathbf{R}$ are intervals. The statement is obviously true if C is one point set. If C is an interval (containing more than one point), then $f(\overline{C}) = \{0,1\} = \overline{\{0,1\}} = \overline{f(C)}$.

Example 3. For the function $f: \mathbf{R}^2 \to \mathbf{R}$,

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

there is a connected subset C of $\mathbf{R}^2$ such that $f(\overline{C}) \subseteq \overline{f(C)}$ is not satisfied, and it follows the function is not connected and it is not path-connected. For example the image of the half line $y = x$, $x \geq 0$ is a two point set $\{0, \frac{1}{2}\}$.

If we remove the origin from the line we obtain a path connected set C such that $f(\overline{C}) = \{0, \frac{1}{2}\}$ and $\overline{f(C)} = \{\frac{1}{2}\}$.

Let X be a metric space.

Definition. If $p, q \in X$, an *interpolation* $T = \{x_0, x_1, \ldots, x_k, x_{k+1}\}$ from p to q is a finite sequence of points

$$p = x_0, x_1, \ldots, x_k, x_{k+1} = q$$

The step of the interpolation is

$$h(T) = \max\{d(x_i, x_{i+1}) \mid i = 0, 1, 2, \ldots, k\}.$$

Definition. Let X and Y be continuums. A function $f: X \to Y$ is a **NS** - *function* (non separating function) if for any two points $p, q \in X$, the following condition holds:

For a sequence of interpolations $T_1 \subseteq T_2 \subseteq T_3 \subseteq \cdots$ between p and q, such that $h(T_i) \to 0$ there exist sequences $(y_i \mid i \in \mathbf{N})$ and $(z_i \mid i \in \mathbf{N}$ in $\bigcup_{i=1}^{\infty} T_i \setminus \{p, q\}$ such that $f(y_i) \to f(p)$ and $f(z_i) \to f(q)$.

Any continuous map is a NS – function.

Theorem 2. If $C \subseteq X$, C connected and $f: X \to Y$ is a NS – function then $f(\overline{C}) \subseteq \overline{f(C)}$.

Proof. If $q \in \overline{C}$, then for a point $p \in \overline{C}$, we choose a sequence of interpolations between p and q, $T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots$ such that $h(T_i) \to 0$. There exists a sequence $(y_i \mid i \in \mathbf{N})$ in $\bigcup_{i=1}^{\infty} T_i \setminus \{p, q\}$ such that $f(y_i) \to f(q)$ and it follows $f(q) \in \overline{f(C)}$.

Example 4. There exists a connected function which is not a NS – function. The function of Cesaro $\omega: [0, 1] \to [0, 1]$ defined with

$$\omega(x) = \lim\, sup \frac{a_1 + \cdots + a_n}{n}, \quad \text{for} \quad 0 \leq x \leq 1,$$

and $x = 0, a_1 \ldots a_n$ is a dyadic expression of x (if x has two expressions we take the finite expression).

We choose a sequence of interpolations

$T_1 = \{0; 0,1; 1\}$,

$T_2 = \{0; 0,01; 011; 1\}$,

$T_3 = \{0; 0,001; 0,01; 0,011; 0,1; 0,101; 0,11; 0,111; 1\}$,

... ...

Then for $y \in T_i$, $y \neq 1$ we have $\omega(y) = 0$. We conclude that the function is not a NS – function.

Example 5. The function $f: \mathbf{R}^2 \to \mathbf{R}$ from Example 3, is continuous by x-coordinate and by y-coordinate (i.e., for a fixed y the function $f_y(x) = f(x,y)$ is continuous and for a fixed x the function $f_x(y) = f(x,y)$ is continuous), and it follows it is path-connected by both coordinates. However the function is not path-connected.

Theorem 3. Let X, Y, Z be Hausdorff spaces and let $f: X \times Y \to Z$ be a function. If f is path-connected, then the following two properties hold for f:

(i) $f(\{x\} \times B)$ is path-connected for each $x \in X$ and each path-connected set $B \subseteq Y$;

(ii) $f(A \times \{y\})$ is path-connected for each path-connected set $A \subseteq X$ and each $y \in Y$.

Proof.

(i) Let $k(t)$ be a path from (x, y_0) to (x, y_1), where $x \in X, y_0, y_1 \in B$. Then $f\big(x, k(t)\big)$ is path in $f(\{x\} \times B)$ from $f(x, y_0)$ to $f(x, y_1)$;

(ii) Similarly.

Theorem 4. Let X, Y, Z be Hausdorff spaces and let $f: X \times Y \to Z$ be a function. Let the conditions (i) and (ii) from Theorem 3 hold. Then $f(A \times B)$ is path-connected for each path-connected subset $A \subseteq X$ and each path-connected subset $B \subseteq Y$.

Proof. Let $(a_0, b_0), (a_1, b_1) \in A \times B$. Since (i) holds, there exists a path k from $f(a_0, b_0)$ to $f(a_0, b_1)$. Since (ii) holds, there exists a path l from $f(a_0, b_1)$ to $f(a_1, b_1)$. Then $h = k * l$ is a path from $f(a_0, b_0)$ to $f(a_1, b_1)$.

Definition. The function $f: X \to Y$, where X and Y are Hausdorff spaces, has *a removable discontinuity at a point* $x \in X$, if for any sequence of points (x_n) in X so that $x_n \neq x$, for each $n \in \mathbf{N}$ and $\lim\limits_{n\to\infty} x_n = x$, there exists $\lim\limits_{n\to\infty} f(x_n)$.

Theorem 5. Let $f: X \to Y$ be path-connected function, X is locally path-connected, connected, 1-countable Hausdorff space and Y is Hausdorff. Then f is continuous at $x \in X$ if and only if f has a removable discontinuity at x.

Proof. Let f has a removable discontinuity at x and let (x_n) be an arbitrary sequence of points in X different from x, that converges to x. If $\lim_{n\to\infty} f(x_n) = f(x)$, then since X is 1-countable it follows that f is continuous. So, we assume that there exists a sequence (x_n), $y = \lim_{n\to\infty} f(x_n)$, and $y \neq f(x)$. Since Y is Hausdorff, there exist neighbourhoods U_1 and V_1 such that $f(x) \in U_1$, $y \in V_1$ and $U_1 \cap V_1 = \emptyset$. Since X is 1-countable, there exists an open set U in X such that $x \in U$ and for each $x' \in U \backslash \{x\}$ it holds $f(x') \in V_1$. From $U_1 \cap V_1 = \emptyset$ it follows $f(x') \notin U_1$. From X being locally path-connected, it follows that there exists an open path-connected set $C \subseteq U$ that contains the point x. The set $f(C)$ is path-connected in Y and $f(x) \in f(C)$. But, $f(C) \cap U_1 = \{f(x)\}$, since for each $x' \in U \setminus \{x\}$ it holds $f(x') \notin U_1$. It follows that $f(x)$ is not a limit point for $f(C)$. Since $f(C)$ is connected, it follows that $f(C) = \{f(x)\}$. So, f is continuous.

If f is continous in x, then for any sequence of points (x_n) in X such that $x_n \neq x$, for each $n \in \mathbf{N}$ and $\lim_{n\to\infty} x_n = x$, we have $f(x) = \lim_{n\to\infty} f(x_n)$, i.e., there exists $\lim_{n\to\infty} f(x_n)$, so f has a removable discontinuity at x.

Definition. Let X and Y be Hausdorff spaces, and $f: X \to Y$ be a function. For each $p \in X$, the set of all $q \in Y$ for which there exists a sequence (p_n) of points in X that converges to p, and $\big(f(p_n)\big)$ converges to q, is called the set of limit points of f at p. This set is denoted by $L(f,p)$.

Theorem 6. Let f be a function from 1-countable Hausdorff space X to 1-countable Hausdorff space Y. Then, for each $p \in X$ the set $L(f,p)$ is closed subset of Y.

Proof. [5], Lemma 3.1.

Theorem 7. Let f be a path-connected function from locally path-connected 1-countable Hausdorff space X to a compact 1-countable Hausdorff space Y. Then $L(f,p)$ is connected subset of Y, for each $p \in X$.

Proof. Let's notice first that for each $p \in X$, the set $L(f,p)$ is not empty, since $f(p) \in L(f,p)$.

Let's make the opposite assumption, i.e. that $L(f,p)$ is not connected for some $p \in X$. Then, there exist non-empty, separated in Y sets A and B such that $L(f,p) = A \cup B$. The sets A and B are closed in $L(f,p)$. By Theorem 6, $L(f,p)$ is closed in Y, so it follows A and B are closed in Y. But, Y is compact, so it is normal. Therefore there exist open in Y sets O_1 and O_2 such that $A \subseteq O_1$, $B \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$.

Let's assume that for each open set O that contains p, there exists $p' \in O$ so that $f(p') \in Y \setminus (O_1 \cup O_2)$. Since X is 1-countable, there exists a decreasing sequence of open sets (O_n) that containg p. As in the proof of Theorem 6, there exists a sequence (p_n) such that (p_n) converges to p, $p_n \in O_n$, $f(p_n) \in Y \setminus (O_1 \cup O_2)$, for each $n \in \mathbf{N}$. The set $Y \setminus (O_1 \cup O_2)$ is closed in Y, therefore compact, since Y is compact. So the set $\{f(p_n) \mid n \in \mathbf{N}\} \subseteq Y \setminus (O_1 \cup O_2)$ has an accumulation point q in $Y \setminus (O_1 \cup O_2)$. Therefore, there exists a subsequence from $\big(f(p_n)\big)$ that converges to q, so $q \in L(f,p)$. It contradicts $L(f,p) \subseteq O_1 \cup O_2$.

Therefore, there exists an open set $O \subseteq X$ such that $f(O) \subseteq O_1 \cup O_2$.

Since X is locally path-connected, there exists path-connected open set $C \subseteq O$ that contains p. The map f is path-connected, so $f(C)$ is path-connected in Y and $f(C) \subseteq O_1 \cup O_2$. Therefore $f(C) \subseteq O_1$ or $f(C) \subseteq O_2$.

Let us assume that $f(C) \subseteq O_1$. If $L(f,p) \cap O_2 \neq \emptyset$ and $y \in L(f,p) \cap O_2$, then there exists a sequence (x_n) that converges to p and $\big(f(x_n)\big)$ converges to y. Since $p \in C$ and C is open, there exists $n_0 \in \mathbf{N}$ so that $x_n \in C$, for each $n \geq n_0$. Then $f(x_n) \in f(C) \subseteq O_1$, for each $n \geq n_0$. Therefore, there exists a neighbourhood O_2 of y in which there are at most a finite number of points from $\big(f(x_n)\big)$, so $\big(f(x_n)\big)$ does not converge to y.

So, if $f(C) \subseteq O_1$, then $L(f,p)$ and O_2 have an empty intersection. Similarly, if $f(C) \subseteq O_2$, the sets $L(f,p)$ and O_1 have an empty intersection, so one of the sets A or B is empty. Therefore we get to a contradiction. It follows that $L(f,p)$ is connected.

Theorem 8. Let f be a path-connected function from locally path-connected 1-countable Hausdorff space X to a compact 1-countable space Hausdorff Y. Then f is continuous at p if and only if $L(f,p)$ is finite or countable.

Proof. Analogously to the proof of Theorem 3.8 from [5].

Corollary. Let f satisfy the conditions of Theorem 7. If Y is metric space and $L(f,p)$ is locally connected, then any two points from $L(f,p)$ can be connected by an arc in $L(f,p)$.

Proof. It follows from [2], page 116, Theorem 3-15.

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ФУНКЦИИ КОИ СЕ СВРЗАНИ ИЛИ ПАТ СВРЗАНИ

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Резиме

Докажани се неколку теореми и дадени се примери за класите на функции кои се сврзани и класите на функции кои се пат сврзани.

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