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DYNAMICAL SYSTEM ON TOPOLOGICAL STRUCTURE OF SET $F(S \times T, X)$

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Abstract

In this paper is proved that, if T is a Hausdorff topological abelian group, and if Y is a Hausdorff topological space, then the mapping $\varphi(f,t) = f_1$, defines a general and continuous dynamical system on topological structure of set $F(S \times T, X)$, relative to the relative topology of coordinatewise convergence.

1. Preliminary remarks

Let T be a Hausdorff topological abelian group, $(Y, \mathcal{V})$ Hausdorff topological space, and $p_s: Y^T \to Y$, $s \in T$ natural projection:

$$\forall f \in Y^T, \quad p_s(f) = f(s), \quad \forall s \in T$$

$$p_s^{-1}(V) = \{f \in Y^T / f(s) \in V\}, \; s \in T, V \in \mathcal{V}.$$

If we denote:

$$\mathcal{S} = \bigcup_{t \in T} \{p_s^{-1}(V) / V \in \mathcal{V}\} = \{p_s^{-1}(V) / s \in T, V \in \mathcal{V}\},$$

then, the family $\mathcal{S}$ forms a subbase for a topology $\mathcal{U} \subset P(X)$ where $X = Y^T$. $\mathcal{U}$ – is the topology of coordinatewise convergence on T. The ordering pair $(X, \mathcal{U})$ – is a Hausdorff topological space. Let $S \subset X$ and if we denote $\mathcal{T}$ – the relative topology of coordinatewise convergence on S, then the ordering pair $(S, \mathcal{T})$ - is also Hausdorff topological space. The set $X^{S \times T}$, we can endow with the topology of coordinatewise convergence on $S \times T$. Indeed, if we denote:

$$p_{(x,t)}: X^{S \times T} \to X$$

$$\forall f \in X^{S\times T}, p_{(x,t)}(f) = F(x,t) = f_{(x,t)}, \forall (x,t) \in S \times T$$
$$p^{-1}_{(x,t)}(U) = \{f \in X^{S\times T}/p_{(x,t)}(f) \in U\}, (x,t) \in S \times T, U \in \mathcal{U}$$
$$f \in p^{-1}_{(x,t)}(U) \iff (P_{(x,t)}(f) \in U, (x,t) \in S \times T, U \in \mathcal{U})$$
$$U = p_s^{-1}(V) = \{f(x,t) \in Y^T/p_s(f(x,t)) = f_s(x,t) \in V\}, s \in T, (x,t) \in S \times T, V \in \mathcal{V}$$
$$M = \{f \in X^{S\times T}/f_s(x,t) \in V\}, s \in T, (x,t) \in S \times T, V \in \mathcal{V}$$
$$\mathcal{H} = \bigcup\{M/M \in \mathcal{M}\},$$

then, the family $\mathcal{H}$ forms a subbase for the $\mathcal{M} \subset P(X^{S\times T})$ topology of coordinatewise convergence on $S \times T$. The ordering pair $(X^{S\times T}, \mathcal{M})$ - is a Hausdorff space. Let $F(S \times T, X) \subset X^{S\times T}$ be the set of all continuous mappings $f: S \times T \to X$ on a topological Hausdorff space $S \times T$ to a topological Hausdorff space X and $i: F(S \times T, X) \to X^{S\times T}$ (the inclusion mapping). The subset $F(S\times T, X)$, we can endow with the relative topology of coordinatewise convergence on $S \times T$. Indeed, if we denote:

$$A = i^{-1}(M) = M \cap F(S \times T, X) = \{f \in X^{S\times T}/f \in M \cap F(S \times T, X)\}$$
$$\mathcal{Q} = \{i^{-1}(M)/M \in \mathcal{M}\}$$

then, the family $\mathcal{Q}$ forms a subbase for the $\mathcal{P} \subset P(F(S \times T, X))$ relative topology of coordinatewise convergence on $S \times T$, and

$$A \in \mathcal{P} \iff A = \{f \in M \cap F(S \times T, X)/f_s(x,t) \in V\}, \quad s \in T, (x,t) \in S \times T, V \in \mathcal{V}, M \in \mathcal{M}.$$

Definition. Let H be a topological space and G any algebraic group. The mapping $\psi: H \times G \to H$ is said to be *a general dynamical system* on H, if satisfying the following axioms:

(a_1) (*Identity property*)
$\psi(x,0) = x, \ \forall x \in H$
(where 0 is the identity of G).

(a_2) (*Group property*)
$\psi(\psi(x,t),s) = \psi(x, t \oplus s), \forall x \in H \& \forall t, s \in G$.
(where $\oplus$ is the group operation of G).

(a_3) (*Continuity property*)
The mapping $\psi: H \times G \to H$ is continuous in H. In other words, for each neighborhood N of point $\psi(x,t)$, there exist a neighborhood E of $x \in H$ such that $\psi(E,t) \subseteq N$.

The general dinamical system $\psi: H \times G \to H$ on H, is said to be a continuous dynamical system on H, if G is a toplogical group and ψ is continuous in $H \times G$.

2. The result

Let $F(S \times T, X)$ be a set of all continuous mappings $f\colon S \times T \to X$. The set $F(S \times T, X)$ has the $\mathcal{P} \subset P(F(S \times T, X))$ relative topology of coordinatewise $S \times T$ convergence on $S \times T$. If the mapping $\varphi\colon F \times T \to F$ defined by:

$$\forall (f,t) \in S \times T, \varphi(f,t) = f_t$$

where $f_t(s) = f(t \oplus s)$, $\forall t, s \in T$, then satisfied theorems:

Theorem 1. *Let T be a Hausdorff topological abelian group and Y Hausdorff topological space. The mapping $\varphi(f,t) = f_t$, defines a general dynamical system on topological structure of set $F(S \times T, X)$ relative to the relative topology of coordinatewise convergence.*

Proof. We shall prove the axioms of dynamical system:

(a$_1$) (*Identity property*). By definition

$$\begin{aligned} \varphi(f(s),t) &= f_t(s) = f(t \oplus s), \forall t, s \in T \\ \varphi(f(s),0) &= f_0(s) = f(0 \oplus s) = f(s), \forall s \in T \\ \varphi(f,0) &= f, \forall f \in F(S \times T, X) \end{aligned}$$

(a$_2$) (*Group property*). For each $t, s \in T$ and for each $f \in F(S \times T, X)$. By the definition

$$\begin{aligned} \varphi(f(s),t) &= f_t(s) = f(t \oplus s), \forall t, s \in T\,, \\ (\varphi(\varphi(f(s),t),m) &= \varphi(f_t(s),m) = \varphi(f(t \oplus s),m) = f_m(t \oplus s) \\ f_m(t \oplus s) &= f(t \oplus s \oplus m) = f_{t \oplus s}(m) \\ \varphi(\varphi(f(s)t),m) &= f_{t \oplus s}(m), \forall m \in T \end{aligned}$$

or

$$\left.\begin{aligned} \varphi(\varphi(f,t),s) &= f_{t \oplus s} \\ f_{t \oplus s} &= \varphi(f, t \oplus s) \end{aligned}\right\} \Rightarrow \varphi(\varphi(f,t),s) = \varphi(f, t \oplus s)\,.$$

(a$_3$) (*Continuity property*). Let us now show that the mapping $\varphi\colon F \times T \to F$ is continuous in topological structure of set $F(S \times T, X)$. Assume that the set $F(S \times T, X)$ has the $\mathcal{P}$ relative topology of coordinatewise convergence on $S \times T$. That is

$$\begin{aligned} A \in \mathcal{P} \iff A =&\{f \in M \cap F(S \times T, X) / f_s(x,t) \in V\}, \\ & s \in T, (x,t) \in S \times T, V \in \mathcal{V}, M \in \mathcal{M}\,. \end{aligned}$$

Let $f \in F(S \times T, X)$ be an arbitrary point and let $\mathcal{N}_f$ be the family of all open neighborhood of point f relative to the relative topology of coordinatewise convergence. The family $\mathcal{N}_f$ can be directed with the binary relation $(\leq) \subseteq \mathcal{N}_f \times \mathcal{N}_f$ as follows:

$$\forall A_1, A_2 \in \mathcal{N}_f, A_1 \leq A_2 \iff A_1 \supseteq A_2 .$$

Then, the ordering pair $(\mathcal{N}_f \leq)$ becomes a directed family. Indeed, for two each open neighborhood $A_1, A_2 \in \mathcal{N}_f$ of the point $f \in F(S \times T, X)$, their intersection $A_3 = A_1 \cap A_2 \in \mathcal{N}_f$ is also an open neighborhood of the point f such that: $A_1 \leq A_3$, $A_1 \leq A_3$. The mapping $g\colon \mathcal{N}_f \to F(S \times T, X)$ which is defined by:

$$\forall A \in \mathcal{N}_f, g(A) = g_A$$

defines the net $(g_A, A \in \mathcal{N}_f) \subset F(S \times T, X)$ wich converges at the unique point $f \in F(S \times T, X)$. In other words, there exist an open neighborhood $A_0 \in \mathcal{N}_f$ of the point $f \in F(S \times T, X)$ such that, for each open neighborhood $A \in \mathcal{N}_f$ of the point $f \in F(S \times T, X)$ is fulfilled:

$$A \geq A_0 \;\Rightarrow\; g_A \in A \subseteq A_0$$

relative to the $\mathcal{P} \subset P(F(S \times T, X))$ relative topology of coordinatewise convergence. The point f is unique, because the ordering pair $(F(S \times T, X), \mathcal{P})$, is a Hausdorff space, relative to the relative topology of coordinatewise convergence. For continuity of mapping $\varphi\colon F \times T \to F$ in $F(S \times T, X)$, it will suffice to show that the corresponding net $(\varphi(g_A, t_0),\ A \in \mathcal{N}_f)$, (where $t_0 \in T$ is a fixed point), converges to a unique point $\varphi(f, t_0) \in F(S \times T, X)$ relative to the relative topology of coordinatewise convergence.

Suppose contrary that the corresponding net $(\varphi(g_A, t_0), A \in \mathcal{N}_f) \subset F(S \times T, X)$ converges to a unique point $\varphi(f, t_0) \in F(S \times T, X)$, but the mapping $\varphi\colon F \times T \to F$ is discontinuous at the point $f \in F(S \times T, X)$. In other words, there exist an open neighborhood $A_0 \in \mathcal{N}_{\varphi(f,t_0)}$ of the point $\varphi(f, t_0)$ in $F(S \times T, X)$ such that, for each open neighborhood $A \in \mathcal{N}_f$ in $F(S \times T, X)$ satisfies the condition:

$$\left.\begin{array}{r}(\exists A_0 \in \mathcal{N}_{\varphi(f,t_0)})(\forall A \in \mathcal{N}_f) \\ \varphi(A, t_0) \not\subset A_0\end{array}\right\}$$

On the other hand, the net $\varphi(g_A, t_0), A \in \mathcal{N}_f) \subset F(S \times T, X)$ converges to a unique point $\varphi(f, t_0) \in F(S \times T, X)$ relative to the $\mathcal{P} \subset P(F(S \times T, X))$ topology of coordinatewise convergence. This means that, there exist an open neighborhood $A_0 \in \mathcal{N}_{\varphi(f,t_0)}$ of the point $\varphi(f, t_0) \in F(S \times T, X)$ such that $\varphi(g_A, t_0) \in A_0$. Hence, we have:

$$\varphi(g_A, t_0) \in \varphi(A, t_0) \not\subset A_0 .$$

Consequently,

$$\left.\begin{array}{r}(\exists A_0 \in \mathcal{N}_{\varphi(f,t_0)})(\forall W \in \mathcal{N}_{\varphi(f,t_0)}) \\ W \geq A_0 \;\Rightarrow\; \varphi(A_0, t_0) \notin W \subseteq A_0\end{array}\right\}$$

The last condition, show that the net $(\varphi(g_A, t_0), A \in \mathcal{N}_f) \subset F(S \times T, X)$ does not convergence to a unique point $\varphi(f, t_0) \in F(S \times T, X)$, which is impossible. This contradiction show that the mapping $\varphi: F \times T \to F$ is continuous in $F(S \times T, X)$, relative to the relative topology of coordinatewise convergence. □

Theorem 2. *Let T be a Hausdorff topological abelian group and Y Hausdorff topological space. The mapping $\varphi(f,t) = f_t$, defines a continuous dynamical system on topological structure of set $F(S \times T, X)$ relative to the relative topology of coordinatewise convergence.*

Proof. We shall prove the axioms of dynamical system:
(a_1) (*Identity property*). By definition

$$\varphi(f(s), t) = f_t(s) = f(t \oplus s), \forall t, s \in T$$
$$\varphi(f, (s), 0) = f_0(s) = f(0 \oplus s) = f(s), \forall s \in T$$
$$\varphi(f, 0) = f, \forall f \in F(S \times T, X)$$

(a_2) (*Group property*). For each $t, s \in T$ and for each $f \in F(S \times T, X)$. By the definition

$$\varphi(f(s), t) = f_t(s) = f(t \oplus s), \forall t, s \in T,$$
$$\varphi(\varphi(f(s), t), m) = \varphi(f_t(s), m) = \varphi(f(t \oplus s), m) = f_m(t \oplus s)$$
$$f_m(t \oplus s) = f(t \oplus s \oplus m) = f_{t \oplus s}(m)$$
$$\varphi(\varphi(f(s), t), m) = f_{t \oplus s}(m), \forall m \in T$$

or

$$\left.\begin{aligned} \varphi(\varphi(f, t), s) &= f_{t \oplus s} \\ f_{t \oplus s} &= \varphi(f, t \oplus s) \end{aligned}\right\} \Rightarrow \varphi(\varphi(f, t), s) = \varphi(f, t \oplus s),$$

(a_3) (*Continuity property*). Let us now show that the mapping $\varphi: F \times T \to F$ is continuous in topological structure of set $F \times T$, where $F = F(S \times T, X)$. We know that $(T, \mathcal{U})$ and $(F(S \times T, X)\mathcal{P})$ are Hausdorff topological space. If we denote:

$$p_1: F \times T \to F, \quad p_2: F \times T \to T \quad \text{(natural projections)}$$
$$p_1^{-1}(A) \in m^* \iff A \in \mathcal{P}, \quad p_2^{-1}(B) \in m^* \iff B \in \mathcal{U}$$
$$\mathcal{B} = \{p_1^{-1}(A) \cap p_2^{-1}(B) / A \in \mathcal{P}, B \in \mathcal{U}\}$$

then, the family $\mathcal{B}$ forms a base for the $m^* \subset P(F \times T)$ topology of coordinatewise convergence. The ordering pair $(F \times T, m^*)$ is a Hausdorff topological space, because F and T are Hausdorff topological space. Consequently:

$$H \in m^* \iff H = p_1^{-1}(C) \cap p_2^{-1}(B) = A \times B = \{(f,t)/f \in A,\, t \in B\}.$$

Let $t \in T$ be an arbitrary point and let N_t, be the family of all open neighborhood of point $t \in T$ in Hausdorff topological space $(T,\mathcal{U})$. The family N_t can be directed with the binary relation $(\leq) \subseteq A_t \times A_t$ as follows:

$$\forall B_1, B_2 \in N_t, B_1 \leq B_2 \iff B_1 \supseteq B_2 .$$

Then, the ordering pair $(N_t, \leq)$ becomes a directed family. Indeed, for two each open neighborhood $B_1, B_2 \in N_t$ of the point $t \in T$, their intersection $B_3 = B_1 \cap B_2 \in N_t$ is also an open neighborhood of the point $t \in T$, such that: $B_1 \leq B_3$, $B_2 \leq B_3$. The mapping $u\colon N_t \to T$ which defined by:

$$\forall B \in N_t,\ u(B) = u_B$$

defined a net $(u_B,\ B \in N_t) \subset T$ which converges at the unique point $t \in T$. In other words, there exist an open neighborhood $B_0 \in N_t$ of the point $t \in T$ such that, for each open neighborhood $B \in N_t$ of the point $t \in T$ fulfilled:

$$B \geq B_0 \Rightarrow u_B \in B \subseteq B_0 .$$

Let $(f,t) \in F \times T$ be an arbitrary point and let $\mathcal{R}_{(f,t)}$ be the family of all open neighborhood of point $(f,t) \in F \times T$ relative to the $m^* \subset P(F \times T)$ topology of coordinatewise convergence. If we denote:

$$\mathcal{R}_{(f,t)} = \mathcal{N}_f \times N_t = \{(A,B)/A \in \mathcal{N}_f, B \in N_t\},$$

then, the family $\mathcal{R}_{(f,t)}$ can be directed with the binary relation $(\leq) \subseteq \mathcal{R}_{(f,t)} \times \mathcal{R}_{(f,t)}$ as follows:

$$\forall (A_1,B_1),(A_2,B_2) \in \mathcal{R}_{(f,t)} \iff (A_1 \leq A_2 \& B_1 \leq B_2)$$
$$\iff (A_1 \supseteq A_2 \& B_1 \supseteq B_2).$$

In this case, the ordering pair $(\mathcal{R}_{(f,t)}, \leq)$ becomes a directed family and the mapping $g^*\colon \mathcal{R}_{(f,t)} \to F \times T$ which defined by:

$$\forall (A,B) \in \mathcal{R}_{(f,t)}, g^*(A,B) = (g_A, u_B = g^*{}_{(A,B)}$$

define a net $(g^*{}_{(A,B)}, (A,B) \in \mathcal{R}_{(f,t)}) \subset F \times T$ which converges at the unique point $(f,t) \in F \times T$. In other words, there is an open neighborhood $(A_0,B_0) \in \mathcal{R}_{(f,t)}$ of the point $(f,t) \in F \times T$ such that, for each open neighborhood $(A,B) \in \mathcal{R}_{(f,t)}$ of the point $(f,t) \in F \times T$ fulfilled:

$$(A,B) \geq (A_0, B_0) \;\Rightarrow\; g^*{}_{A,B)} \in A \cap B \subseteq A_0 \cap B_0$$

or

$$\left.\begin{array}{r} (\exists (A_0, B_0) \in \mathcal{R}_{(f,t)})(\forall (A,B) \in \mathcal{R}_{(f,t)}) \\ (A,B) \geq (A_0,B_0) \iff g^*{}_{(A,B)} \in A \cap B \subseteq A_0 \cap B_0 \end{array}\right\}$$

relative to the $m^* \subset P(F \times T)$ topology of coordinatewise convergence. The point $(f,t) \in F \times T$ unique, because the pair $(F \times T, m^*)$, is the Hausdorff space, relative to the m^*- topology of coordinatewise convergence. For continuity of mapping $\varphi\colon F \times T \to F$ in $F \times T$, it will suffice to show that the corresponding net $(\varphi(g_A, u_B), (A,B) \in \mathcal{R}_{(f,t)}) \subset F(S \times T, X)$ converges to a unique point $\varphi(f,t) \in F(S \times T, X)$ relative to the $\mathcal{P} \subset P(F(S \times T, X))$ topology of coordinatewise convergence.

Suppose contrary that the corresponding net $(\varphi(g_A.u_B), (A,B) \in \mathcal{R}_{(f,t)}) \subset F(S \times T, X)$ converges to a unique point $\varphi(f,t) \in F(S \times T, X)$, but the mapping $\varphi\colon F \times T \to F$ is discontinuous at a point $\varphi(f,t) \in F(S \times T, X)$. In other words, there is a neighborhood $A_0 \in \mathcal{N}_{(f,t)}$ in $F(S \times T, X)$ such that, for each open neighborhood $(A,B) \in \mathcal{R}_{(f,t)}$ in $F \times T$, satisfies the condition:

$$\left.\begin{array}{r} (\exists A_0 \in \mathcal{N}_{(f,t)})(\forall (A,B) \in \mathcal{R}_{(f,t)}) \\ \varphi(A,B) \not\subset A_0 \end{array}\right\}$$

On the other hand, the net $(\varphi(g_A, u_B), (A,B) \in \mathcal{R}_{(f,t)}) \subset F(S \times T, X)$ converges to a unique point $\varphi(f,t) \in F(S \times T, X)$ relative to the $\mathcal{P} \subset P(F(S \times T, X))$ topology of coordinatewise convergence. This means that, there exist an open neighborhood $A_0 \in \mathcal{N}_{(f,t)}$ of the point $\varphi(f,t) \in F(S \times T, X)$ in $F(S \times T, X)$ such that $\varphi(g_A, u_B \in A_0$. So, we have:

$$\varphi(g^*{}_{(A,B)}) = \varphi(g_A, u_B) \in \varphi(A,B) \not\subset A_0$$

Consequently,

$$\left.\begin{array}{r} (\exists A_0 \in \mathcal{N}_{(f,t)})(\forall A \in \mathcal{N}_{(f,t)}) \\ A \geq A_0 \;\Rightarrow\; \varphi(g_A, u_B) \notin A \subseteq A_0 \end{array}\right\}$$

The last contition, shows that the net $(\varphi(g_A, u_B), (A,B) \in \mathcal{R}_{(f,t)}) \subset F(S \times T, X)$ does not converge to a unique point $\varphi(f,t) \in F(S \times T, X)$, which is impossible. This contradiction show that the mapping $\varphi\colon F \times T \to F$ is continuous in $F \times T$, relative to the relative topology of coordinatewise convergence. □

References

[1] Sell G. R: *Topological Dynamics and Ordinary Differential Equations,* University of Menneapolis, U. S. A. New York, 1971.

[2] Sibirski K. S.: *Videnie Vtopologi?ekuju dinamiku,* Kišinev, 1970.

ДИНАМИЧКИОТ СИСТЕМ ВО ТОПОЛОШКАТА СТРУКТУРА НА МНОЖЕСТВА $F(S \times T, X)$

Неки Дервиши

Р е з и м е

Во оваа работа е докажано со две теореми дека: Ако T е абелова тополошка група на Хаусдорф, тогаш мапингот $\varphi(f,t) = f_t$ одредува еден динамички генерален и еден динамички непрекинет систем на тополошка структура на множества $F(S \times T, X)$ во врска со релативна топологија на конвергенцијата по координати.

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