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TWO SUBSTITUTIONS IN ONE SPECIAL NO HOMOGENOUS VECUA EQUATION

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Abstract

In this paper two theorems are proved and according to them the special no homogeneous Vecua equation (6) is transformed to the basic Vecua equation (8).

Introduction

The Vecua equation

$$\frac{\hat{d}W}{d\bar{z}} = A(z)W + B(z)\overline{W} + F(z). \tag{1}$$

which is complex writing for the elliptic system of partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 2(a_1 + b_1)u - 2(a_2 - b_2)v + 2f_1 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 2(a_2 + b_2)u - 2(a_1 - b_1)v + 2f_2 \end{cases} \tag{2}$$

defines different classes of so cold generalized analytic functions $W = W(z) = u(x,y) + iv(x,y)$, $z = x + iy$. [1], [2], [3], [4] etc. Here

$$\frac{\hat{d}W}{d\bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \tag{3}$$

is so cold operatory derivative of the function $W = W(z)$ from the variable $\overline{z} = x - iy$ or areolar derivative, introduced by Kolosov [5] in 1909. We should mark that the operatory equation

$$\frac{\hat{d}W}{d\overline{z}} = 0 \tag{4}$$

is a complex writing of the Cauchy - Riemann conditions so, in the class of the functions $W = W(z)$ with real $u(x, y) = \operatorname{Re} W(z)$ and imaginary part $v(x, y) = \operatorname{Im} W(z)$ functions with continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in the area $D \subseteq \mathbf{C}$, it defines analytic functions from $z = x + iy$.

The operating rules for the operatory derivative from $\overline{z} = x - iy$ are fully given in the Положий monograph [2]. In the mentioned monograph is also defined so cold operator integral by $\overline{z}$

$$\hat{\int} f(z)\, dz = F(z) + \Phi(z) \tag{5}$$

where $\frac{\hat{d}F(z)}{d\overline{z}} = f(z)$ in the working area $D \subseteq \mathbf{C}$ i.e. $F = F(z)$ is one operatory primitive function from $\overline{z} = x - iy$ for the function $f = f(z)$ in D and $\frac{\hat{d}\Phi(z)}{d\overline{z}} = 0$ i.e. $\Phi = \Phi(z)$ is arbitrary analytic function from $z = x + iy$ in the role of integral constant.

Until now, in the science literature, for the Vecua equation (1) and for equations

$$\varphi\Big(z, W, \frac{\hat{d}W}{d\overline{z}}, \frac{\hat{d}^2W}{d\overline{z}^2}, \dots, \frac{\hat{d}^nW}{d\overline{z}^n}\Big) = 0$$

where the unknown function $W = W(z)$ is under the sign for complex conjugation there aren't any methods for their solving with quadratures. There for the mathematicians whose work is the above mentioned are forced to look and to discus individual examples and problems, which have at least some general nature. For example such mathematicians are: N.Teodorescu (Romania), Gabrinovic (Russia), M.Balk (Germany), M.Canak N.Ralević, J.Kečkić, S.Fempl (Serbia), B.Martić (Bosnia and Hercegovina), D.Dimitrovski, B.Ilievski, S.Brsakoska (R.Macedonia) and many others.

In this paper we will view the no homogenous Vecua equation

$$\frac{\hat{d}W}{d\overline{z}} = \overline{W} + F(z) \tag{6}$$

Theorem 1. If the free member $F = F(z)$ in the no homogenous Vecua equation (6) is antianalytic function from z i.e. a function which is under complex conjugation of an analytic function from z, then with the substitution

$$\omega = W + \overline{F} \tag{7}$$

the equation (6) is transformed in the corresponding basic Vecua equation

$$\frac{\hat{d}\omega}{d\overline{z}} = \overline{\omega} \tag{8}$$

Proof. From (7), according to the operator rules for the operatory derivative by $\overline{z}$ (3), we have

$$\frac{\hat{d}\omega}{d\overline{z}} = \frac{\hat{d}W}{d\overline{z}} + \frac{\hat{d}\overline{F}}{d\overline{z}},$$

and according to (4) - operatory derivative by $\overline{z}$ from analytic function is zero i.e. $\frac{\hat{d}\overline{F}}{d\overline{z}} = 0$, we have $\frac{\hat{d}\omega}{d\overline{z}} = \frac{\hat{d}W}{d\overline{z}}$.

With the new substitution (7) in the equation (6), previously written in the shape

$$\frac{\hat{d}W}{d\overline{z}} = \overline{W + \overline{F}},$$

we get

$$\frac{\hat{d}\omega}{d\overline{z}} = \overline{\omega} \tag{8}$$

what had to be proved.

Note 1. According to the formulas (10), (11) and (13) in the paper [6], for $\lambda = 1$, for the general solution of the basic Vecua equation (8) we have

$$\omega = \Phi(z) + \sum_{n=0}^{\infty} \frac{\overline{z}^{n+1}}{(n+1)!} \hat{\int}\hat{\int} \cdots \hat{\int}\hat{\int} \Phi(z)\,(dz)^{n+1}$$
$$+ \sum_{n=0}^{\infty} \frac{z^n}{n!} \hat{\int}\hat{\int} \cdots \hat{\int}\hat{\int} \overline{\Phi}(z)(d\overline{z})^{n+1} \tag{9}$$

or

$$\omega = \Phi(z) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!\,n!} \hat{\int} z^{n+1}(z-\zeta)^n \Phi(\zeta)\,d\zeta + \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \hat{\int} z^n(\overline{z}-\overline{\zeta})^n \overline{\Phi}(\zeta)\,d\overline{\zeta} \tag{9'}$$

or

$$\omega = \Phi(z) + \hat{\int} \frac{\hat{d}S}{d\overline{z}} \Phi(\zeta)\,d\zeta + \hat{\int} S\overline{\Phi}(\zeta)\,d\overline{\zeta} \tag{9''}$$

where $S = S(u) = \sum\limits_{n=0}^{\infty} \frac{u^n}{(n!)^2}$, $u = z(\overline{z}-\overline{\zeta})$ and $\Phi = \Phi(z)$ is analytic function in the role of integral constant. According to this and the substitution (7), the general solution of the no homogenous equation (6), is

$$W = \Phi(z) + \sum_{n=0}^{\infty} \frac{\overline{z}^{\,n+1}}{(n+1)!} \hat{\int}\hat{\int} \cdots \hat{\int}\hat{\int} \Phi(z)\,(dz)^{n+1} + \sum_{n=0}^{\infty} \frac{z^n}{n!} \hat{\int}\hat{\int} \cdots \hat{\int}\hat{\int} \overline{\Phi}(z)\,(d\overline{z})^{n+1} - \overline{F} \tag{10}$$

or

$$W = \Phi(z) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!\,n!} \hat{\int} z^{n+1}(z-\zeta)^n \Phi(\zeta)\,d\zeta + \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \hat{\int} z^n(\overline{z}-\overline{\zeta})^n \overline{\Phi}(\zeta)\,d\overline{\zeta} - \overline{F}(z) \tag{10'}$$

or

$$W = \Phi(z) + \hat{\int} \frac{\hat{d}S}{d\overline{z}} \Phi(\zeta)\,d\zeta + \hat{\int} S\overline{\Phi}(\zeta)\,d\overline{\zeta} - \overline{F}(z)\,. \tag{10''}$$

Example 1. The areolar equation

$$\frac{\hat{d}W}{d\overline{z}} = e^{-W}\left(e^{\overline{W}} + e^{\overline{z}}\right),$$

according to

$$e^{\overline{z}} = e^{x-iy} = e^x(\cos y - i\sin y) = \overline{e^x(\cos y + i\sin y)} = \overline{e^z}\,,$$

we mark it as

$$e^W \frac{\hat{d}W}{d\overline{z}} = \overline{e^W} + \overline{e^z}$$

i.e. in the shape

$$\frac{\hat{d}}{d\overline{z}}(e^W) = \overline{e^W} + \overline{e^z}.$$

With the substitution

$$e^W = \Omega \tag{$*$}$$

the last equation is transformed in

$$\frac{\hat{d}\Omega}{d\overline{z}} = \overline{\Omega} + \overline{e^z},$$

which is from the shape (6) from the unknown function $\Omega = \Omega(z)$ and where the free member is no analytic function from z.

With the substitution

$$\omega = \Omega + e^z \tag{$**$}$$

the last equation is transformed in the basic Vecua equation

$$\frac{\hat{d}\omega}{d\overline{z}} = \overline{\omega} \tag{8}$$

whose solution, according to (9″) is

$$\omega = \Phi(z) + \hat{\int} \frac{\hat{d}S}{d\overline{z}} \Phi(\zeta) d\zeta + \hat{\int} S\overline{\Phi}(\zeta)\, d\overline{\zeta}.$$

With the substitutions ($*$) and ($**$), the general solution of the starting areolar equation is

$$e^W = \Phi(z) + \hat{\int} \frac{\hat{d}S}{d\overline{z}} \Phi(\zeta) d\zeta + \hat{\int} S\overline{\Phi}(\zeta)\, d\overline{\zeta} - e^z.$$

Theorem 2. If we know one particular solution of the no homogenous Vecua equation

$$\frac{\hat{d}W}{d\overline{z}} = \overline{W} + F(z) \tag{6}$$

than with the linear substitution

$$W = \omega + W_1 \tag{11}$$

the equation (6) is transformed in a corresponding basic Vecua equation

$$\frac{\hat{d}\omega}{d\overline{z}} = \overline{\omega} \tag{8}$$

from the unknown function $\omega = \omega(z)$.

Proof. From the substitution (11) we get

$$\frac{\hat{d}W}{d\overline{z}} = \frac{\hat{d}\omega}{d\overline{z}} + \frac{\hat{d}W_1}{d\overline{z}} \tag{12}$$

and with changing (11) and (12) in (6) we get

$$\frac{\hat{d}\omega}{d\overline{z}} + \frac{\hat{d}W_1}{d\overline{z}} = \overline{\omega + W_1} + F$$

i.e.

$$\frac{\hat{d}\omega}{d\overline{z}} + \frac{\hat{d}W_1}{d\overline{z}} = \overline{\omega} + \overline{W}_1 + F\,. \tag{13}$$

Using the fact that $W_1 = W_1(z)$ is a solution of the equation (6) i.e. that $\dfrac{\hat{d}W_1}{d\overline{z}} = \overline{W}_1 + F$ in $D \subseteq \mathbf{C}$ the equation (13) will be

$$\frac{\hat{d}\omega}{d\overline{z}} = \overline{\omega} \tag{8}$$

what had to be proved.

Note 2. According to the note 1 i.e. according to the formulas (9), (9′) and (9″) and the substitution (11), the general solution of the no homogenous Vecua equation (6) can be written as a sum of the general solution of the corresponding basic Vecua equation (8) and one particular integral of the no homogenous equation (6).

According to this, the general solution of the equation (6) is

$$\begin{aligned} W = \Phi(z) &+ \sum_{n=0}^{\infty} \frac{\overline{z}^{\,n+1}}{(n+1)\,!} \hat{\int}\hat{\int} \cdots \hat{\int}\hat{\int} \Phi(z)(dz)^{n+1} \\ &+ \sum_{n=0}^{\infty} \frac{z^n}{n\,!} \hat{\int}\hat{\int} \cdots \hat{\int}\hat{\int} \overline{\Phi}(z)(d\overline{z})^{n+1} - W_1 \end{aligned} \tag{14}$$

or

$$\begin{aligned} W = \Phi(z) &+ \sum_{n=0}^{\infty} \frac{1}{(n+1)\,!\,n\,!} \hat{\int} z^{n+1}(z-\zeta)^n \Phi(\zeta) d\zeta \\ &+ \sum_{n=0}^{\infty} \frac{1}{(n\,!)^2} \hat{\int} z^n (\overline{z} - \overline{\zeta})^n \overline{\Phi}(\zeta)\, d\overline{\zeta} + W_1 \end{aligned} \tag{14′}$$

or

$$W = \Phi(z) + \hat{\int} \frac{\hat{d}S}{d\bar{z}} \Phi(\zeta) d\zeta + \hat{\int} S\overline{\Phi}(\zeta) d\bar{\zeta} + W_1 \qquad (14'')$$

where $S = S(z,\zeta) = \sum_{n=0}^{\infty} \frac{\left[z(\bar{z}-\bar{\zeta})\right]^n}{(n!)^2}$.

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ДВЕ СМЕНИ ВО ЕДНА СПЕЦИЈАЛНА НЕХОМОГЕНА РАВЕНКА ВЕКУА

Борко Илиевски, Слаѓана Брсакоска и Петар Соколоски

Р е з и м е

Во овој труд се докажани две теореми според кои специјалната нехомогена равенка Векуа (6) се трансформира во основната равенка Векуа (8).

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