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EXACT SOLUTIONS TO A FAMILY OF FITZHUGH-NAGUMO-TYPE EQUATIONS

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Abstract

A family of partial differential equations of type Fitzhugh-Nagumo is studied in this paper, generalizing a result presented in [7]. After an introduction to the problem, we find the exact solutions of the equation under study. The first integral method is employed.

1. Introduction

Partial differential equations of type FitzHugh–Nagumo bear the name from FitzHugh and Nagumo who first studied equations of this type in [5], [9] as a simplified nerve conduction model. Such nonlinear reaction—diffusion equations attracted great interest lately because of its potential application in Medicine. For example, it was used as a mathematical model of cardiac excitation [1]. It is also met in circuit theory, biology and genetics [2]. Another mathematical model related to the heart problems has been proposed in [10]. It is a FitzHugh–Nagumo type system. Recently, the authors in [6] investigate the Tzitzeica—Dodd—Bullough partial differential equation of the form $u_{xt} = e^{-u} + e^{-2u}$, using the Exp-method, pointing out new periodic solutions.

The partial differential equation considered in this paper is given by:

$$u_t - u_{xx} = Mu^3 + Nu^2 + Pu + Q \tag{1}$$

with $M, N, P, Q \in \mathbf{R}$.

When $M = -1, N = Q = 0, P = 1$, the equation (1) becomes the real Newell—Whitehead equation and for $M = -1, N = \alpha + 1, P = -\alpha, Q = 0$, $\alpha \in \mathbf{R}$, it falls on the classical FitzHugh–Nagumo equation [7].

Information about some exact solutions of (1) can be found in [11]. The aim of this paper is to obtain another exact solutions, namely travelling-wave solutions, for (1) using the first integral method. This method was used by the authors in [3], [4] for searching solutions of other nonlinear partial differential equations.

2. Exact Solutions

In order to proceed to the investigation of the equation (1), we look for solutions of the form $u(x,t) = u(\eta)$, with $\eta = x + \beta t$, $\beta \in \mathbf{R}$. Replacing $u(x,t) = u(x + \beta t) = U(\eta)$ in (1) one gets

$$\beta U' - U'' = MU^3 + NU^2 + PU + Q. \tag{2}$$

Denoting $U' = V$, from (2) a system of differential equations of first order is obtained:

$$\begin{aligned} U' &= V \\ V' &= \beta V - (MU^3 + NU^2 + PU + Q). \end{aligned} \tag{3}$$

In order to solve (3) the first integral method will be used. For a nontrivial solution $(U(\eta), V(\eta))$ of (3) we search a first integral of (3) in the form $f(U,V) = a_0(U) - V$. Then

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial U}\frac{\partial U}{\partial \eta} + \frac{\partial f}{\partial V}\frac{\partial V}{\partial \eta} = 0 \tag{4}$$

leads to

$$a_0'(U) \cdot V - \beta V + MU^3 + NU^2 + PU + Q = 0. \tag{5}$$

The condition $f(U,V) = K$ leads to $V = a_0(U) - K$. In the following computation the constant K does not play a special role, so we will consider $K = 0$, i.e. $V = a_0(U)$.

From (5) the following relation is obtained

$$a_0(U)\left(a_0'(U)-\beta\right)+MU^3+NU^2+PU+Q=0. \tag{6}$$

By identifying the coefficients of U in (6) it results that $a_0(U)$ must be a polynomial of second degree, $a_0(U)=AU^2+BU+C$ whose coefficients must satisfy

$$\begin{cases} 2A^2+M=0 \\ 3AB-A\beta+N=0 \\ 2AC+B^2-B\beta+P=0 \\ BC-C\beta+Q=0\,. \end{cases} \tag{7}$$

Let us denote $M=-2S^2, S>0$.

The unknowns of (7) are A, B, C and β.

By replacing $A=S$, $S>0$ (similar computation can be made for $A=-S$), respectively B and C as function of β in the last relation (7) one can determine β from the equation

$$4S^3\beta^3-3\beta S\left(N^2+6PS^2\right)-54QS^4-9NPS^2-N^3=0. \tag{8}$$

The equation (8) has at least a real solution.

For a solution $\beta_1\in\mathbf{R}$ of (8) the coefficients of $a_0(U)$ are given by

$$\begin{aligned} A=&S,\\ B=&\frac{1}{3S}\left(-N+S\beta\right),\\ C=&-\frac{1}{18S^3}\left(NS\beta+N^2+9PS^2-2S^2\beta^2\right). \end{aligned} \tag{9}$$

In order to determine the travelling wave solution of (1) the relation $f(U,V)=0$ will be used. It is an ordinary differential equation of first degree:

$$U'=AU^2+BU+C. \tag{10}$$

We consider first the case

$$\Delta:=B^2-4\cdot A\cdot C=\frac{-2S^2\beta_1^2-12S^2P+2N^2}{9A^2}>0,$$

i.e. $\beta_1^2<\dfrac{N^2-6S^2P}{S^2}$ (do not forget that $S>0$). In this situation the equation $AU^2+BU+C=0$ has two real solutions α_1 and α_2. The equation (10) becomes

$$U' = A(U - \alpha_1)(U - \alpha_2).$$

Its general solution is

$$U(\eta) = \frac{\alpha_2 - K \cdot \alpha_1 \cdot e^{A(\alpha_2 - \alpha_1)\eta}}{1 - K \cdot e^{A(\alpha_2 - \alpha_1)\eta}}. \tag{11}$$

In order to define the solution for every $x \in \mathbf{R}$ and for every $t \geq 0$ the condition $K < 0$ must be imposed. By considering $K = -\lambda^2$ we obtain the solution

$$u(x,t) = \frac{\alpha_2 + \lambda^2 \cdot \alpha_1 \cdot e^{A(\alpha_2 - \alpha_1)(x + \beta_1 t)}}{1 + \lambda^2 \cdot e^{A(\alpha_2 - \alpha_1)(x + \beta_1 t)}}. \tag{12}$$

In the case $\Delta = 0$, the equation $AU^2 + BU + C = 0$ has a double real solution $\alpha = -\frac{B}{2A}$ and the equation (10) has the general solution

$$U(\eta) = \frac{-1}{A\eta + K} - \frac{B}{2A}. \tag{13}$$

There are not values of K for which the solution is defined for every $x \in \mathbf{R}$ and for every $t \geq 0$, hence in this situation some spatial constraint are imposed, namely $x < -K - \beta t$ or $x > -K - \beta t$.

For $\Delta < 0$, i.e. $\beta_1^2 > \frac{N^2 - 6S^2P}{S^2}$ the general solution of (10) is

$$U(\eta) = \sqrt{p} \tan \sqrt{p}(A\eta + K) - \frac{B}{2A}, \tag{14}$$

with $p = -\frac{B^2 - 4AC}{4A^2} > 0$.

The solution

$$u(x,t) = \sqrt{p} \tan \sqrt{p}(A(x + \beta t) + K) - \frac{B}{2A}$$

can not express the action of a dynamical system because it can not be defined for every $x \in \mathbf{R}$ and $t \geq 0$.

Remark 1. *Another method for searching exact solutions for equations of the form* $u_t - u_{xx} = -(u - u_1)(u - u_2)(u - u_3), u_1 < u_2 < u_3$ *and* $u_1, u_2, u_3 \in \mathbf{R}$ *is presented in* [8]. *Our approach in this paper allows to find exact solutions for such equations even though* u_1, u_2, u_3 *are complex numbers* (Example 2), *which is not presented in* [8].

Example 1. Let us consider a specific case

$$u_t - u_{xx} = Nu^2 - u^3,$$

corresponding to $M = -1$, $\left(S = 1/\sqrt{2}\right)$, $P = 0$, $Q = 0$ and arbitrary $N \neq 0$ in (1) .

The equation (8) becomes

$$\beta^3 - \frac{3}{2} N^2 \beta - \frac{\sqrt{2}}{2} N^3 = 0.$$

Its solutions are $\beta_1 = \beta_2 = -\dfrac{N\sqrt{2}}{2}$ and $\beta_3 = N\sqrt{2}$.

In the case $\beta_1 = -\dfrac{N\sqrt{2}}{2}$ the values the coefficients involved in the computation are $A = \dfrac{\sqrt{2}}{2}$, $B = -\dfrac{N\sqrt{2}}{2}$, $C = 0$ and $\Delta = B^2 > 0$. In this case $\alpha_1 = 0$ and $\alpha_2 = N$ and (12) gives the analytical form of an exact solution:

$$u(x,t) = \frac{N}{1 + \lambda^2 \cdot e^{\frac{\sqrt{2}}{2} N \left(x - \frac{N\sqrt{2}}{2} t\right)}}.$$

For $\beta = N\sqrt{2}$ we obtain $A = \dfrac{\sqrt{2}}{2}$, $B = 0$ and $C = 0$. In this case $\Delta = 0$ and an other exact solution is given by (13):

$$u(x,t) = \frac{-2}{\sqrt{2}x + 2Nt + 2K}. \tag{15}$$

For a given K the solution is defined on

$$D_1 = \left\{(x,t) \quad | \quad t \geq 0 \ , \ x < N\sqrt{2}t - K\right\}$$

or on

$$D_2 = \left\{(x,t) \quad | \quad t \geq 0 \ , \ x > N\sqrt{2}t - K\right\}.$$

Example 2. Let us consider now another case corresponding to $M = -2$, $(S = 1)$, $N = P = 0$, $Q = -2$. Eq (1) becomes

$$u_t - u_{xx} = -2u^3 - 2.$$

In this case (8) leads to $\beta^3 + 27 = 0$, having the real root $\beta = -3$. We have also $A \mp 1, B = -1, C = 1$. From (14), the solution is given by:

$$u(x,t) = \sqrt{p}\tan\sqrt{p}\,(A(x+\beta t)+K) - \frac{B}{2A}$$

i.e.

$$u(x,t) = \frac{\sqrt{3}}{2}\tan\frac{\sqrt{3}}{2}(K - 3t + x) + \frac{1}{2}.$$

3. Conclusions

In this work we investigated a family of partial differential equations of type Fitzhugh—Nagumo, with a general cubic function on the right hand side of the equation, generalizing a result presented in [7]. Exact solutions of this family of equations are pointed out. As applications are presented two examples.

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ТОЧНИ РЕШЕНИЈА ЗА СЕМЕЈСТВОТО НА FITZHUGH-NAGUMO-ВИДОТ РАВЕНКИ

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Р е з и м е

Во овој текст се проучува семејството на делумни диференцијални равенки од видот Fitzhugh-Nagumo користејќи го резултатот претставен во [7]. По воведот во проблемот, ги наоѓаме точните решенија на равенката која се применува. Се применува првиот интегрален метод.

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