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ON THE EXISTENCE OF A TYPE OF RICCI-RECURRENT MANIFOLD

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Abstract

In the present paper we study properties of Ricci-recurrent manifolds which satisfy certain conditions, and prove the existence of such manifolds by a non-trivial concrete example.

1. Introduction

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be Ricci-recurrent [4] if its Ricci tensor S of type $(0, 2)$ is *not* proportional to the metric tensor g and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z), \tag{1.1}$$

for all vector fields X, Y, Z, where A is a non-zero 1-form and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Ricci-recurrent spaces have been studied by M. C. Chaki [1], W. Roter [5,6,7] and others.

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be a quasi Einstein manifold if the Ricci tensor S satisfies the condition ([2]):

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1.2}$$

where a and b are non-zero scalars and A is a non-zero 1-form.

The paper is organized as follows:

Considering a symmetric endomorphism L of the tangent space at each point of (M^n, g) corresponding to the Ricci tensor S, i.e.,

$$g(LX, Y) = S(X, Y),$$

we study the existence of Ricci-recurrent manifolds (M^n, g) of non-zero scalar curvature r, which satisfy the condition

$$L^2X = \frac{r}{n-1} LX. \tag{1.3}$$

At first it is shown that if such a manifold exists, then it must be of dimension three and L will have only two eigenvalues 0 and $\frac{r}{2}$, of which the former is simple and the latter is of multiplicity two. We also prove that such a Ricci-recurrent manifold is a quasi Einstein manifold in the sense of Chaki and Maity ([2]). Then a manifold (M^3, g) is constructed, whose metric in local coordinates $(x^i)_{i=\overline{1,3}}$ is given by

$$ds^2 = e^{(x^1+x^2)^2}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2. \tag{1.4}$$

It is shown that the manifold with this metric provides the existence of such a Ricci-recurrent manifold.

2. Main results

We suppose that $(M^n, g)(n > 2)$ satisfies the conditions (1.1) and (1.3). From (1.1) we get

$$(\operatorname{div} L)(X) = A(LX), \tag{2.1}$$

where 'div' denotes the divergence. Again transvecting (1.1) we get

$$dr(X) = X.r = A(X)r. \tag{2.2}$$

From Bianchi's second identity it follows that

$$2(\operatorname{div} L)(X) = dr(X). \tag{2.3}$$

Hence from (2.2) and (2.3) we obtain

$$(\operatorname{div} L)(LX) = \frac{r}{2}A(LX). \tag{2.4}$$

Putting LX for X in (2.1) and using (1.3) yields

$$(\operatorname{div} L)(LX) = \frac{r}{n-1}A(LX). \tag{2.5}$$

From (2.4) and (2.5) we get

$$rA(LX)\frac{3-n}{2(n-1)} = 0,$$

from which it follows that $n = 3$, since $r \neq 0$, $A \neq 0$. We can therefore state the following result:

Theorem 2.1. *If a Ricci-recurrent manifold satisfies the condition* (1.3), *then it must be of dimension three.*

Let X be an eigenvector corresponding to the eigenvalue λ of L. Then $LX = \lambda X$. Hence, considering the relation $L^2X = \frac{r}{2}LX$, we get

$$\left(\lambda - \frac{r}{2}\right)\lambda X = 0.$$

So $\lambda \in \{0, \frac{r}{2}\}$. Let ρ be the associated vector field corresponding to the 1-form A. That is, $g(X, \rho) = A(X)$. Then from (2.1), (2.2) and (2.3) it follows that

$$A(LX) = \frac{r}{2}A(X).$$

Therefore, $g(LX,\rho) = g(\frac{r}{2}X,\rho)$ for all vector fields X, which implies that

$$L\rho = \frac{r}{2}\rho. \tag{2.6}$$

From (2.6) we conclude that ρ is an eigenvector corresponding to the eigenvalue $\frac{r}{2}$ of L. Summing up, we can state the following result:

Theorem 2.2. *If a Ricci-recurrent manifold (M^3, g) satisfies the condition $L^2X = \frac{r}{2}LX$, then L will have only two eigenvalues, namely 0 and $\frac{r}{2}$, of which the former is simple and the latter is of multiplicity 2. Further, if ρ is the vector corresponding to the 1-form A, then ρ is the eigenvector corresponding to the eigenvalue $\frac{r}{2}$ of L.*

From Th. 2.1 it follows that a Ricci-recurrent manifold $(M^n, g)(n > 2)$ satisfying (1.3) is of dimension three. Hence the manifold is conformally flat. That is, $C = 0$ where C is the Weyl conformal curvature tensor defined by

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)LX - g(X,Z)LY + S(Y,Z)X \\ &- S(X,Z)Y] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

R denotes the curvature tensor of type $(1,3)$. Since $C = 0$, therefore $\operatorname{div} C = 0$ where 'div' denotes divergence. Hence it follows that [3]

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Y)dr(Z)].$$

For $n = 3$, the above relation becomes

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = \frac{1}{4}[g(Y,Z)dr(X) - g(X,Y)dr(Z)]. \tag{2.7}$$

Using (1.1) and (2.2) in (2.7) it follows that

$$A(X)[S(Y,Z) - \frac{r}{4}g(Y,Z)] = A(Z)\left[S(Y,X) - \frac{r}{4}g(Y,X)\right]$$

or,

$$A(X)T(Y,Z) = A(Z)T(Y,X), \tag{2.8}$$

where

$$T(Y,Z) = S(Y,Z) - \frac{r}{4}g(Y,Z). \tag{2.9}$$

Replacing $X = \rho$ in (2.8) we obtain

$$T(Y,Z) = \frac{A(Z)}{A(\rho)}T(Y,\rho) = \frac{A(Z)}{A(\rho)}\left[S(Y,\rho) - \frac{r}{4}g(Y,\rho)\right]. \tag{2.10}$$

Using (2.6) in (2.10) yields

$$T(Y,Z) = \frac{rA(Z)}{4A(\rho)}A(Y). \tag{2.11}$$

Therefore using (2.9) in (2.11) we obtain

$$S(Y,Z) = \frac{r}{4}g(Y,Z) + \frac{r}{4A(\rho)}A(Y)A(Z). \tag{2.12}$$

Hence from (1.2) and (2.12) we can state the following:

Theorem 2.3. *A Ricci-recurrent manifold* $(M^n, g)(n > 2)$ *satisfying* (1.3) *is a quasi Einstein manifold.*

Finally, suppose that in the Ricci-recurrent manifold under consideration the vector field ρ defined by $g(X,\rho) = A(X)$ is a parallel vector field, that is, $\nabla_X \rho = 0$ for all X. Then by the Ricci identity we get

$$R(X,Y)\rho = 0. \tag{2.6}$$

Hence $S(X,\rho) = 0$. From (2.6) it follows that $S(X,\rho) = \frac{r}{2}g(X,\rho)$. Hence $r = 0$, which is a contradiction. This leads to the following:

Theorem 2.4. *In a Ricci-recurrent manifold satisfying* (1.3), *the assciated vector field* ρ *can not be a parallel vector field.*

3. The existence of a Ricci-recurrent (M^3, g) satisfying (1.3)

In this section we prove the existence of a Ricci-recurrent (M^3, g) satisfying the condition $L^2X = \frac{r}{n-1}LX$ by constructing a non-trivial concrete example.

Example. Let us consider a Riemannian metric g on $\mathbf{R}^3$ by

$$ds^2 = g_{ij}dx^i dx^j = e^{(x^1+x^2)^2}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2, \tag{3.1}$$

$(i, j = 1, 2, 3)$. Then the only non-vanishing components of the Christoffel symbols and of the curvature tensor are

$$\Gamma^1_{11} = -(x^1 + x^2)e^{(x^1+x^2)^2}, \qquad \Gamma^2_{11} = (x^1+x^2)e^{(x^1+x^2)^2} + (x^1+x^2)e^{2(x^1+x^2)^2},$$

$$\Gamma^2_{12} = \Gamma^2_{21} = (x^1+x^2)e^{(x^1+x^2)^2}, \quad R_{1221} = [2(x^1+x^2)^2 + 1]e^{(x^1+x^2)^2}$$

and the components obtained by the symmetry properties. The non-vanishing components of R^i_j and their derivatives are given by

$$R^1_1 = R^2_2 = -[2(x^1+x^2)^2 + 1]e^{(x^1+x^2)^2}. \tag{3.2}$$

$$R^1_{1,1} = R^1_{1,2} = R^2_{2,1} = R^2_{2,2} = -2(x^1+x^2)[2(x^1+x^2)^2 + 3]e^{(x^1+x^2)^2}. \tag{3.3}$$

From (3.2) we get that the scalar curvature r of the resulting space $(\mathbf{R}^3, g)$ is $r = -2[2(x^1+x^2)^2+1]e^{(x^1+x^2)^2}$, which is non-vanishing and non-constant.

Also from (3.2) it can be easily verified that $R^i_j R^j_k = \frac{r}{2} R^i_k$ which implies that relation (1.3) is satisfied. We shall now show that $\mathbf{R}^3$ is a Ricci-recurrent space. Let us consider the associated 1-forms as follows:

$$\lambda_1 = \lambda_2 = \frac{2(x^1+x^2)[2(x^1+x^2)^2+3]}{[2(x^1+x^2)^2+1]} . \tag{3.4}$$

Since if A is a 1-form then $A(X)$ is a scalar and for being a function of x^1, x^2, the right side of (3.4) is a scalar; hence λ_1, λ_2 are 1-forms. To verify the relation (1.1), it is sufficient to check the following:

$$\begin{aligned} &\text{(i) } R^1_{1,1} = \lambda_1 R^1_1, \quad &&\text{(ii) } R^1_{1,2} = \lambda_2 R^1_1, \\ &\text{(iii) } R^2_{2,1} = \lambda_1 R^2_2, \quad &&\text{(iv) } R^2_{2,2} = \lambda_2 R^2_2 , \end{aligned} \tag{3.5}$$

since for the other cases (1.1) trivially holds. By (3.2) and (3.4) we get:

$$\begin{aligned} \lambda_1 R^1_1 &= -\frac{2(x^1+x^2)[2(x^1+x^2)^2+3]}{[2(x^1+x^2)^2+1]}[2(x^1+x^2)^2+1]e^{(x^1+x^2)^2} \\ &= -2(x^1+x^2)[2(x^1+x^2)^2+3]e^{(x^1+x^2)^2} = R^1_{1,1} , \end{aligned}$$

which proves (3.5)-(i). By a similar argument it can be shown that (ii), (iii) and (iv) of (3.5) are also true. Hence $\mathbf{R}^3$ equipped with the metric g, given in (3.1), is a Ricci-recurrent space which satisfies the relation (1.3). Thus we can state the following:

Theorem 3.1. *Let* $(\mathbf{R}^3, g)$ *be a Riemannian space endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = e^{(x^1+x^2)^2}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2,$$

$(i,j = 1,2,3)$. *Then* $(\mathbf{R}^3, g)$ *is a Ricci-recurrent space which satisfies the relation* (1.3) *and whose scalar curvature is non-zero and non-constant.*

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ЗА ПОСТОЕЊЕТО НА ВИДОТ RICCI-ПОВТОРНА РАЗНОВИДНОСТ

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Р е з и м е

Во овој текст ние ги проучуваме особините на Ricci-повторна разновидност кои задоволуваат одредени услови и го докажуваат постоењето на такви разновидности преку необични конкретни примери.

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