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EQUIVALENCE OF THE FINITE COVER AND THE MINIMAL COVER PROPERTIES

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Abstract. The notion of a cover appears in topology, in connection with compact spaces, and in lattice theory in connection with compact elements. In the first case, covers are defined as particular families of open sets, and, in the second case they are defined as subsets of a lattice. Compact subsets of a topological space and compact elements in a lattice have the property , each of their covers to be reducible to a finite cover. In this paper the notion of a minimal cover is introduced as a cover that contains no proper subcover, and it is proved that the compact subsets in a topological space and the compact elements in a lattice are exactly those that have the property, each of their covers to contain a minimal subcover.

1. FCP AND MCP

Let X be a non-empty set and $\mathcal{F} = \{U_i : i \in I\}$ be a class of subsets of X. Consider a subclass $\mathcal{A}$ of $\mathcal{F}$ such that $A \subseteq \bigcup \mathcal{A}$ for some $A \subseteq X$. We call $\mathcal{A}$ a cover of A contained in $\mathcal{F}$. Furthermore, every subclass of $\mathcal{A}$ which is a cover of A in its own right is called a subcover of $\mathcal{A}$. If a finite subclass of $\mathcal{A}$ is a cover of A, i.e. if

$$\exists U_{i_1}, U_{i_2}, \ldots, U_{i_m} \in \mathcal{A} \text{ such that } A \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_m}$$

then $\mathcal{A}$ is said to be reducible to a finite cover.

If $\mathcal{A}$ contains no proper subcover, then $\mathcal{A}$ is said to be a minimal cover. A subcover of $\mathcal{A}$ is said to be a minimal subcover if it is a minimal cover.

Definition 1.1. *A subset A of a non-empty set X satisfies the finite cover property (FCP) with respect to a class $\mathcal{F}$ of subsets of X if every cover of A contained in $\mathcal{F}$ is reducible to a finite cover.*

Definition 1.2. *A subset A of a non-empty set X satisfies the minimal cover property (MCP) with respect to a class $\mathcal{F}$ of subsets of X if every cover of A contained in $\mathcal{F}$ contains a minimal subcover.*

Proposition 1.1. *If a subset A of a non-empty set X satisfies (FCP) with respect to a class $\mathcal{F}$ of subsets of X, then A satisfies (MCP) with respect to $\mathcal{F}$.*

Proof. Let A be a subset of X, and let $\mathcal{A}$ be a cover of A contained in $\mathcal{F}$. By definition of (FCP), $\mathcal{A}$ contains a finite subcover. Every finite cover that is not

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minimal can be reduced to a smaller cover, and therefore with a finite number of steps every finite cover can be reduced to a minimal subcover. □

Observe that in general (MCP) does not imply (FCP).

Example. Consider the class $\mathcal{F} = \{[i, i+1] : i \in \mathbb{Z}\}$ of subsets of the set of real numbers $\mathbb{R}$. $\mathcal{F}$ is a cover of $\mathbb{R}$, moreover it is the only cover of $\mathbb{R}$ contained in $\mathcal{F}$. Thus $\mathcal{F}$ is a minimal cover of $\mathbb{R}$, and $\mathbb{R}$ satisfies the minimal cover property with respect to $\mathcal{F}$. However, $\mathcal{F}$ cannot be reduced to a finite cover, therefore $\mathbb{R}$ does not satisfy the finite cover property.

We will say that a class of sets $\mathcal{F}$ is closed under finite union if the union of any finite number of sets in $\mathcal{F}$ is also in $\mathcal{F}$.

Proposition 1.2. *Let X be a non-empty set, $A \subseteq X$, and let $\mathcal{F}$ be a class of subsets of X closed under finite union. Then the following statements are equivalent:*
(i) A satisfies (FCP) with respect to $\mathcal{F}$,
(ii) A satisfies (MCP) with respect to $\mathcal{F}$.

Proof. Suppose that every cover of A contained in $\mathcal{F}$ contains a minimal subcover, and suppose that A does not satisfy the finite cover property with respect to $\mathcal{F}$. Then there is a cover $\mathcal{A}$ of A contained in $\mathcal{F}$ that contains no finite subcover. Let $\mathcal{U} = \{U_i : i \in I\}$ be a minimal subcover of $\mathcal{A}$. $\mathcal{U}$ is not finite, and therefore there exists a denumerable subset $J = \{i_1, i_2, \dots\}$ of I. Let,

$$\begin{aligned} V_i &= U_i, \text{ for } i \in I \setminus J, \\ V_{i_1} &= U_{i_1}, \\ V_{i_n} &= V_{i_{n-1}} \cup U_{i_n}, \text{ for } n > 1 \end{aligned}$$

For each $i \in I$, $V_i \in \mathcal{F}$, since $\mathcal{F}$ is closed under finite union. Also, $\bigcup_{i\in I} V_i = \bigcup_{i \in I} U_i$. Hence, $\mathcal{V} = \{V_i : i \in I\}$ is a cover of A contained in $\mathcal{F}$. For each $n, m \in \mathbb{N}$, $n < m$ implies $V_{i_n} \subset V_{i_m}$. Therefore every minimal subcover of $\mathcal{V}$ contains at most one set of $\{V_{i_n} : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$,

$$\left(\bigcup_{i \in I\setminus J} V_i\right) \cup V_{i_n} \subseteq \bigcup_{i \in I \setminus \{i_{n+1}\}} U_i.$$

By the choice of $\mathcal{U} = \{U_i : i \in I\}$ it follows that there is no $n \in \mathbb{N}$ such that $\mathcal{W}_n = \{V_i : i \in I \setminus J\} \cup \{V_{i_n}\}$ is a cover of A. Since every minimal subcover of $\mathcal{V}$ must be contained in some $\mathcal{W}_n$, it follows that $\mathcal{V}$ containes no minimal subcover, and this contradicts the supposition that every cover of A contained in $\mathcal{F}$ contains minimal subcover. □

2. Topological Spaces

Subsets of a topological space that satisfy the finite cover property with respect to the topology are called compact. The covers contained in the topology are called open covers. Since every topology is closed under arbitrary union, we get the following special case of Proposition 1.2..

Proposition 2.1. *A subset A of a topological space X is compact if and only if every open cover of A contains a minimal subcover.*

3. Lattices

In this section, it is shown that a similar result to the one of Proposition 2.2 can be proved for compact elements of a lattice.

Definition 3.1. *Let L be a lattice. A subset A of L is called a cover of an element a in L if $\bigvee A$ exists and $a \leq \bigvee A$. Furthermore, if a finite subset of A is also a cover of a, i.e. if*

$$\exists\, a_1, ..., a_n \in A \text{ such that } a \leq \bigvee \{a_1, ..., a_n\}$$

then A is said to be reducible to a finite cover, or contains a finite subcover.

If A contains no proper subset B such that $\bigvee B$ exists and $a \leq \bigvee B$, then A is said to be a minimal cover. A subset of A is said to be a minimal subcover if it is a minimal cover.

Definition 3.2. *An element a in a lattice L is compact if every cover of a is reducible to a finite cover.*

Proposition 3.1. *Let L be a lattice. An element a in L is compact if and only if every cover of a contains a minimal subcover.*

Proof. Let a be a compact element in L, and let A be a cover of a. By the definition for a compact element, A contains a finite subcover. Every finite cover that is not minimal can be reduced to a smaller cover, and therefore with a finite number of steps every finite cover can be reduced to a minimal cover.

To prove the implication in the other direction suppose that every cover of a containes a minimal subcover, and suppose that a is not compact. Then there is a cover A of a that contains no finite subcover. Let $B = \{a_i : i \in I\}$ be a minimal subcover of A. B cannot be finite and therefore there exists a denumerable subset $J = \{i_1, i_2, ...\}$ of I. Let,

$$b_i = a_i, \text{ for } i \in I \setminus J, \qquad b_{i_1} = a_{i_1}, b_{i_n} = b_{i_{n-1}} \vee a_{i_n}, \text{ for } n > 1$$

Observe that $\bigvee_{i \in I} b_i$ exists and $\bigvee_{i \in I} b_i = \bigvee_{i \in I} a_i$. Hence, $C = \{b_i : i \in I\}$ is a cover of a. For each $n, m \in \mathbb{N}$, $n < m$ implies $b_{i_n} < b_{i_m}$. Therefore every minimal subcover of C contains at most one element from $\{b_j : j \in J\}$. If for some $n \in \mathbb{N}$, there is a minimal subcover of C contained in $\{b_i : i \in I \backslash J\} \cup \{b_{i_n}\}$, then it can be shown without difficulty that there is a subcover of B contained in

$$\{a_i : i \in I \backslash \{i_{n+1}\}\}$$

and that contradicts the supposition that B is a minimal subcover. □

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