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GENERALIZATIONS OF STEFFENSEN'S INEQUALITY VIA n WEIGHT FUNCTIONS

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Abstract. New generalizations of Steffensen's inequality are obtained by means of weighted Montgomery identity with n different weight functions. Instead for a nondecreasing (1-convex) function our generalization hold for a n-convex function. Further, functionals associated to these new generalizations are observed and used to generate $n-$exponentially and exponentially convex functions as well as to obtain new Stolarsky type means related to these functionals.

1. Introduction

The well-known Steffensen's inequality states (see [15])

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be integrable mappings on $[a,b]$ such that f is nonincreasing and $0\leq g(t)\leq 1$ for $t\in[a,b]$. Then*

$$\int_{b-\lambda}^{b} f(t)\,dt \leq \int_{a}^{b} f(t)\,g(t)\,dt \leq \int_{a}^{a+\lambda} f(t)\,dt \tag{1.1}$$

where $\lambda=\int_a^b g(t)\,dt$.

J. F. Steffensen proved this inequality in 1918 and since then it was generalized in numerous ways. Extensive overview of these generalizations can be found in [10] or [14].

In the recent paper [3] the next weighted Euler identity is obtained:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be n-times differentiable on $[a,b]\,, n\in\mathbb{N}$ with $f^{(n)}$ $:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Let $w_i:[a,b]\to[0,\infty\rangle$, $i=1,..,n$ be a sequence of n integrable functions satisfying $\int_a^b w_i(t)\,dt=1$ and $W_i(t)=\int_a^t w_i(x)\,dx$ for*

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$t \in [a,b]$, $W_i(t) = 0$ for $t < a$ and $W_i(t) = 1$ for $t > b$, for all $i = 1,..,n$. For any $x \in [a,b]$ define weighted Peano kernel:

$$P_{w_i}(x,t) = \begin{cases} W_i(t), & a \le t \le x, \\ W_i(t) - 1 & x < t \le b. \end{cases}$$

Then it holds

$$\begin{aligned} & f(x) - \int_a^b w_1(t) f(t)\,dt - \sum_{k=0}^{n-2} \left(\int_a^b w_{k+2}(t) f^{(k+1)}(t)\,dt \right) \\ & \cdot \left(\int_a^b \cdots \int_a^b P_{w_1}(x,t_1) \prod_{i=1}^{k} P_{w_{i+1}}(t_i,t_{i+1})\,dt_1 \cdots dt_{k+1} \right) \\ & = \int_a^b \cdots \int_a^b P_{w_1}(x,t_1) \prod_{i=1}^{n-1} P_{w_{i+1}}(t_i,t_{i+1}) f^{(n)}(t_n)\,dt_1 \cdots dt_n. \end{aligned} \tag{1.2}$$

If we take $w_i \equiv w$, $i = 1,..,n$ identity (1.2) reduces to identity obtained in [1], and for $n = 1$, it reduces to the **weighted Montgomery identity** given by Pečarić in [11]

$$f(x) - \int_a^b w_1(t) f(t)\,dt = \int_a^b P_{w_1}(x,t_1) f'(t_1)\,dt_1.$$

The aim of this paper is to generalize Steffensen's inequality by using the weighted Euler identity (1.2). In a such way new generalizations Steffensen's inequality for a n-convex function are obtained in Section 2 and Section 3. In case $n = 1$ Steffensen's inequality (1.1) is recaptured since 1-convex functions are monotonic (nondecreasing) functions. In such way we generalize for any $n \in \mathbb{N}$ results obtained in [6] for $n = 1$. In Section 4 estimates of the difference of the left-hand and right-hand sides of the obtained inequalities are given. In Section 5, three functionals associated to these new generalizations are considered and used to generate $n-$exponentially and exponentially convex functions. In Section 6, new Stolarsky type means related to these functionals are obtained.

2. The difference between two weighted integral means

Next, we subtract two generalized weighted Montgomery identities (1.2) to obtain identity for the difference between two weighted integral means, each having its own weight, on two different intersecting intervals $[a,b]$ and $[c,d]$. This identity is given in the next theorem for both possible cases, when one interval is a subset of the other $[c,d] \subseteq [a,b]$ and for overlapping intervals $[a,b] \cap [c,d] = [c,b]$. The other two possible cases, when $[a,b] \cap [c,d] \neq \emptyset$ we simply get by replacement $a \leftrightarrow c$, $b \leftrightarrow d$. For that purpose we denote

$$T_{w_1,..,w_n}^{[a,b]}(x) = \sum_{k=0}^{n-2} \left(\frac{1}{\int_a^b w_{k+2}(t)\,dt} \int_a^b w_{k+2}(t) f^{(k+1)}(t)\,dt \right)$$

$$\cdot\left(\int_a^b\cdots\int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{k}P_{w_{i+1}}(t_i,t_{i+1})\,dt_1\cdots dt_{k+1}\right).$$

Theorem 3. *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be n-times differentiable on $[a,b]\cup[c,d]$, $n\in\mathbb{N}$ with $f^{(n)}:[a,b]\cup[c,d]\to\mathbb{R}$ integrable on $[a,b]\cup[c,d]$. Let $w_i:[a,b]\to[0,\infty\rangle$, $i=1,..,n$ be a sequence of n integrable functions, $W_i(t)=\int_a^t w_i(x)\,dx$ for $t\in[a,b]$, $W_i(t)=0$ for $t<a$ and $W_i(t)=\int_a^b w_i(x)\,dx$ for $t>b$, for all $i=1,..,n$. Also, let $u_i:[c,d]\to[0,\infty\rangle$, $i=1,..,n$ be a sequence of n integrable functions $U_i(t)=\int_c^t u_i(x)\,dx$ for $t\in[c,d]$, $U_i(t)=0$ for $t<c$ and $U_i(t)=\int_c^d u_i(x)\,dx$ for $t>d$, for all $i=1,..,n$. For any $x\in[a,b]\cup[c,d]$ define weighted Peano kernel:*

$$P_{w_i}(x,t)=\begin{cases}\frac{1}{W_i(b)}W_i(t), & a\le t\le x,\\ \frac{1}{W_i(b)}W_i(t)-1, & x<t\le b,\\ 0, & t\notin[a,b],\end{cases}$$

$$P_{u_i}(x,t)=\begin{cases}\frac{1}{U_i(d)}U_i(t), & c\le t\le x,\\ \frac{1}{U_i(d)}U_i(t)-1, & x<t\le d,\\ 0, & t\notin[c,d].\end{cases}$$

Then if $W_i(b)\neq 0$ and $U_i(d)\neq 0$ for $i=1,..,n$, for any $x\in[a,b]\cap[c,d]$ it holds

$$\frac{1}{\int_c^d u_1(t)dt}\int_c^d u_1(t)f(t)\,dt-\frac{1}{\int_a^b w_1(t)dt}\int_a^b w_1(t)f(t)\,dt--T^{[a,b]}_{w_1,..,w_n}(x)+T^{[c,d]}_{u_1,..,u_n}(x)= \\ =\int_{\min\{a,c\}}^{\max\{b,d\}}K(x,t_1,\ldots,t_n)\,f^{(n)}(t_n)\,dt_n \tag{2.1}$$

where

$$K(x,t_1,\ldots,t_n)=\int_{\min\{a,c\}}^{\max\{b,d\}}\cdots\int_{\min\{a,c\}}^{\max\{b,d\}}\left[P_{w_1}(x,t_1)\prod_{i=1}^{n-1}P_{w_{i+1}}(t_i,t_{i+1})\right. \tag{2.2}$$
$$\left.-P_{u_1}(x,t_1)\prod_{i=1}^{n-1}P_{u_{i+1}}(t_i,t_{i+1})\right]dt_1\cdots dt_{n-1}$$

Proof. We apply (1.2) with $x\in[a,b]\cap[c,d]$ and n normalized weight functions $w_i(t)/W_i(b)$, $t\in[a,b]$, $i=1,..,n$ and then once again with n normalized weight functions $u_i(t)/U_i(d)$, $t\in[c,d]$, $i=1,..,n$. By subtracting these two identities we obtain

$$\frac{1}{\int_c^d u_1(t)\,dt}\int_c^d u_1(t)f(t)\,dt-\frac{1}{\int_a^b w_1(t)\,dt}\int_a^b w_1(t)f(t)\,dt-T^{[a,b]}_{w_1,..,w_n}(x)+T^{[c,d]}_{u_1,..,u_n}(x)$$
$$=\int_a^b\cdots\int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{n-1}P_{w_{i+1}}(t_i,t_{i+1})f^{(n)}(t_n)\,dt_1\cdots dt_n$$
$$-\int_c^d\cdots\int_c^d P_{u_1}(x,t_1)\prod_{i=1}^{n-1}P_{u_{i+1}}(t_i,t_{i+1})f^{(n)}(t_n)\,dt_1\cdots dt_n$$

$$= \int_{\min\{a,c\}}^{\max\{b,d\}} K(x, t_1, \ldots, t_n) f^{(n)}(t_n)\, dt_n$$

and (2.1) is proved. □

Consider the sequence $(B_k(t), k \geq 0)$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1, \quad B_0(t) = 1$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$

The values $B_k = B_k(0)$, $k \geq 0$ are known as Bernoulli numbers. For our purposes, the first five Bernoulli polynomials are

$$B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t + \frac{1}{6},$$
$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\ B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}.$$

Let $(B_k^*(t), k \geq 0)$ be a sequence of periodic functions of period 1, related to Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1, \qquad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}.$$

From the properties of Bernoulli polynomials it easily follows that $B_0^*(t) = 1, B_1^*$ is discontinuous function with a jump of -1 at each integer, while B_k^*, $k \geq 2$, are continuous functions.

Corollary 3.1. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be n-times differentiable on $[a,b] \cup [c,d]$, $n \in \mathbb{N}$ with $f^{(n)} : [a,b] \cup [c,d] \to \mathbb{R}$ integrable on $[a,b] \cup [c,d]$. Let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ be integrable weight functions, $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = \int_a^b w(x)\,dx$ for $t > b$, $U(t) = \int_c^t u(x)\,dx$ for $t \in [c,d]$, $U(t) = 0$ for $t < c$ and $U(t) = \int_c^d u(x)\,dx$ for $t > d$. Then if $W(b) \neq 0$ and $U(d) \neq 0$, for any $x \in [a,b] \cap [c,d]$ it holds*

$$\begin{aligned}
&\frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) f(t)\,dt - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - T_w^{[a,b]}(x) + T_u^{[c,d]}(x) \\
&= \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b P_w(x,s) \left[B_{n-1}\left(\frac{s-a}{b-a}\right) - B_{n-1}^*\left(\frac{s-t}{b-a}\right) \right] ds \right) f^{(n)}(t)\,dt \\
&- \frac{(d-c)^{n-2}}{(n-1)!} \int_c^d \left(\int_c^d P_u(x,s) \left[B_{n-1}\left(\frac{s-c}{d-c}\right) - B_{n-1}^*\left(\frac{s-t}{d-c}\right) \right] ds \right) f^{(n)}(t)\,dt
\end{aligned} \tag{2.3}$$

where

$$T_w^{[a,b]}(x) = \sum_{k=0}^{n-2} \frac{(b-a)^{k-1}}{k!} \left(\int_a^b P_w(x,t) B_k\left(\frac{t-a}{b-a}\right) dt \right) \left(f^{(k)}(b) - f^{(k)}(a) \right)$$

$$T_u^{[c,d]}(x)=\sum_{k=0}^{n-2}\frac{(d-c)^{k-1}}{k!}\left(\int_c^d P_w(x,t)\,B_k\left(\frac{t-c}{d-c}\right)dt\right)\left(f^{(k)}(d)-f^{(k)}(c)\right)$$

Proof. We apply identity (2.1) with $w_1\equiv w$, $w_i\equiv\frac{1}{b-a}$, $i=2,..,n$ and $u_1\equiv u$, $u_i\equiv\frac{1}{d-c}$, $i=2,..,n$. Then $P_{w_i}(x,t)$ and $P_{u_i}(x,t)$ for $i=2,..,n$ reduce to

$$P_{a,b}(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\le t\le x,\\ \frac{t-b}{b-a}, & x<t\le b,\\ 0, & t\notin[a,b].\end{cases}\quad\text{and}\quad P_{c,d}(x,t)=\begin{cases}\frac{t-c}{d-c}, & c\le t\le x,\\ \frac{t-d}{d-c}, & x<t\le d,\\ 0, & t\notin[c,d].\end{cases}$$

Since the the next two identities hold (see [4])

$$\int_a^b\cdots\int_a^b P_{a,b}(x,s_1)\left(\prod_{i=1}^{k-1}P_{a,b}(s_i,s_{i+1})\right)ds_1\cdots ds_k=\frac{(b-a)^k}{k!}B_k\left(\frac{x-a}{b-a}\right)$$

and

$$\int_a^b\cdots\int_a^b P_{a,b}(x,s_1)\left(\prod_{i=1}^{n-2}P_{a,b}(s_i,s_{i+1})\right)ds_1\cdots ds_{n-2}$$
$$=\frac{(b-a)^{n-2}}{(n-1)!}\left[B_{n-1}\left(\frac{x-a}{b-a}\right)-B^*_{n-1}\left(\frac{x-s_n}{b-a}\right)\right]$$

it follows that

$$\frac{1}{b-a}\int_a^b\cdots\int_a^b P_w(x,t_1)\prod_{i=1}^{k}P_{a,b}(t_i,t_{i+1})\,dt_1\cdots dt_{k+1}$$
$$=\frac{(b-a)^{k-1}}{k!}\left(\int_a^b P_w(x,t)\,B_k\left(\frac{t-a}{b-a}\right)dt\right)$$

and

$$\int_a^b\cdots\int_a^b P_w(x,t_1)\prod_{i=1}^{n-1}P_{a,b}(t_i,t_{i+1})\,f^{(n)}(t_n)\,dt_1\cdots dt_n$$
$$=\frac{(b-a)^{n-2}}{(n-1)!}\int_a^b\left(\int_a^b P_w(x,s)\left[B_{n-1}\left(\frac{s-a}{b-a}\right)-B^*_{n-1}\left(\frac{s-t}{b-a}\right)\right]ds\right)f^{(n)}(t)\,dt.$$

Consequently $T^{[a,b]}_{w_1,..,w_n}(x)$ reduces to

$$T_w^{[a,b]}(x)$$
$$=\tfrac{1}{b-a}\textstyle\sum_{k=0}^{n-2}\left(\int_a^b\cdots\int_a^b P_w(x,t_1)\prod_{i=1}^{k}P_{a,b}(t_i,t_{i+1})\,dt_1\cdots dt_{k+1}\right)\left(f^{(k)}(b)-f^{(k)}(a)\right)$$
$$=\textstyle\sum_{k=0}^{n-2}\frac{(b-a)^{k-1}}{k!}\left(\int_a^b P_w(x,t)\,B_k\left(\frac{t-a}{b-a}\right)dt\right)\left(f^{(k)}(b)-f^{(k)}(a)\right)$$

and similarly $T^{[c,d]}_{u_1,..,u_n}(x)$ to $T_u^{[c,d]}(x)$. Finally

$$\int_{\min\{a,c\}}^{\max\{b,d\}}K(x,t_1,\ldots,t_n)\,f^{(n)}(t_n)\,dt_n$$

$$= \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b P_w(x,s) \left[B_{n-1}\left(\frac{s-a}{b-a}\right) - B_{n-1}^*\left(\frac{s-t}{b-a}\right) \right] ds \right) f^{(n)}(t)\,dt$$

$$- \frac{(d-c)^{n-2}}{(n-1)!} \int_c^d \left(\int_c^d P_u(x,s) \left[B_{n-1}\left(\frac{s-c}{d-c}\right) - B_{n-1}^*\left(\frac{s-t}{d-c}\right) \right] ds \right) f^{(n)}(t)\,dt$$

and identity (2.1) reduces to identity (2.3). □

Corollary 3.2. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be n-times differentiable on $[a,b] \cup [c,d]$, $n \in \mathbb{N}$ with $f^{(n)} : [a,b] \cup [c,d] \to \mathbb{R}$ integrable on $[a,b] \cup [c,d]$. Let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ be integrable weight functions, $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = \int_a^b w(x)\,dx$ for $t > b$, $U(t) = \int_c^t u(x)\,dx$ for $t \in [c,d]$, $U(t) = 0$ for $t < c$ and $U(t) = \int_c^d u(x)\,dx$ for $t > d$. Then if $W(b) \neq 0$ and $U(d) \neq 0$, for any $x \in [a,b] \cap [c,d]$ it holds*

$$\frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) f(t)\,dt - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x)$$
$$= \int_{\min\{a,c\}}^{\max\{b,d\}} \widehat{K}(x,t_1,\ldots,t_n) f^{(n)}(t_n)\,dt_n \tag{2.4}$$

where

$$T_{w,n}^{[a,b]}(x) = \sum_{k=0}^{n-2} \left(\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f^{(k+1)}(t)\,dt \right) \cdot \left(\int_a^b \cdots \int_a^b P_w(x,t_1) \prod_{i=1}^{k} P_w(t_i,t_{i+1})\,dt_1 \cdots dt_{k+1} \right),$$

$$T_{u,n}^{[c,d]}(x) = \sum_{k=0}^{n-2} \left(\frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) f^{(k+1)}(t)\,dt \right) \cdot \left(\int_c^d \cdots \int_c^d P_u(x,t_1) \prod_{i=1}^{k} P_u(t_i,t_{i+1})\,dt_1 \cdots dt_{k+1} \right),$$

and

$$\widehat{K}(x,t_1,\ldots,t_n) = \int_{\min\{a,c\}}^{\max\{b,d\}} \cdots \int_{\min\{a,c\}}^{\max\{b,d\}} \left[P_w(x,t_1) \prod_{i=1}^{n-1} P_w(t_i,t_{i+1}) \right.$$
$$= \left. -P_u(x,t_1) \prod_{i=1}^{n-1} P_u(t_i,t_{i+1}) \right] dt_1 \cdots dt_{n-1}$$

Proof. We apply identity (2.1) with $w_i \equiv w$, $i = 1,..,n$. Then $T_{w_1,..,w_n}^{[a,b]}(x)$, $T_{u_1,..,u_n}^{[c,d]}(x)$ and $K(x,t_1,\ldots,t_n)$ reduce to $T_{w,n}^{[a,b]}(x)$, $T_{u,n}^{[c,d]}(x)$ and $\widehat{K}(x,t_1,\ldots,t_n)$ respectively. □

Remark 2.1: Identity (2.3) was previously obtained in [2]. In a special case for uniform normalized weight function w, for the case $[c,d] \subseteq [a,b]$ it was obtained in [12] and for the case $[a,b] \cap [c,d] = [c,b]$ in [5]. Identity (2.4), in a special case for uniform normalized weight function w, for $c = d$ as a limit case and $n = 2$ was obtained in [1], while for $n = 3$ it was obtained in [4].

Theorem 4. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be n-convex function on $[a,b] \cup [c,d]$, $n \in \mathbb{N}$. Let $w_i : [a,b] \to [0,\infty\rangle$, $i = 1,..,n$ be a sequence of n integrable functions, $W_i(t) = \int_a^t w_i(x)\,dx$ for $t \in [a,b]$, $W_i(t) = 0$ for $t < a$ and $W_i(t) = \int_a^b w_i(x)\,dx$ for $t > b$,for all $i = 1,..,n$. Also, let $u_i : [c,d] \to [0,\infty\rangle$, $i = 1,..,n$ be a sequence of n integrable functions, $U_i(t) = \int_c^t u_i(x)\,dx$ for $t \in [c,d]$, $U_i(t) = 0$ for $t < c$ and $U_i(t) = \int_c^d u_i(x)\,dx$ for $t > d$, for all $i = 1,..,n$. If*

$$K \geq 0 \tag{2.5}$$

where K is the function defined by (2.2), then for any $x \in [a,b] \cap [c,d]$ it holds

$$\frac{1}{\int_a^b w_1(t)\,dt}\int_a^b w_1(t) f(t)\,dt + T_{w_1,..,w_n}^{[a,b]}(x) \leq \frac{1}{\int_c^d u_1(t)\,dt}\int_c^d u_1(t) f(t)\,dt + T_{u_1,..,u_n}^{[c,d]}(x)\,. \tag{2.6}$$

Proof. Since f is a n-convex function, without loss of generality we can assume (see [14, p. 293]) that $f^{(n)}$ exists and is continuous. By using the (2.1) and $f^{(n)} \geq 0$ the proof follows. □

Remark 2.2: Inequality (2.6) holds also if f is n-concave and $K \leq 0$. If f is n-concave and $K \geq 0$ or f is n-convex and $K \leq 0$ the inequality (2.6) is reversed.

3. Generalizations of Steffensen's inequality

Corollary 4.1. *Let $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ be n-convex function on $[a,b] \cup [a,a+\lambda]$, $n \in \mathbb{N}$. Let $w_i : [a,b] \to [0,\infty\rangle$, $i = 1,..,n$ and $u_i : [a,a+\lambda] \to [0,\infty\rangle$, $i = 1,..,n$ be two sequences of weight functions as in Theorem 3. If*

$$K \geq 0 \tag{3.1}$$

where K is the function defined by (2.2), then for any $x \in [a,b] \cap [a,a+\lambda]$ it holds:

$$\begin{aligned} &\frac{1}{\int_a^b w_1(t)dt}\int_a^b w_1(t) f(t)\,dt + T_{w_1,..,w_n}^{[a,b]}(x) \leq \\ &\leq \frac{1}{\int_a^{a+\lambda} u_1(t)dt}\int_a^{a+\lambda} u_1(t) f(t)\,dt + T_{u_1,..,u_n}^{[a,a+\lambda]}(x)\,. \end{aligned} \tag{3.2}$$

In case f is n-concave function inequality (3.2) holds if $K \leq 0$.

Proof. We apply Theorem 4 with $[c,d] = [a,a+\lambda]$. □

Remark 3.1: For every differentiable, nonincreasing function $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ and $w : [a,b] \to [0,\infty\rangle$ and $u : [a,a+\lambda] \to [0,\infty\rangle$ some weight functions such that $\int_a^b w(t)\,dt = \int_a^{a+\lambda} u(t)\,dt$ inequality (3.2) for $n = 1$ reduces to

$$\int_a^b w(t) f(t)\,dt \leq \int_a^{a+\lambda} u(t) f(t)\,dt$$

while condition $K \leq 0$ reduces to

$$\int_a^x u(t)\,dt \geq \int_a^x w(t)\,dt \text{ for } x \in [a, a+\lambda] \text{ and } \int_x^b w(t)\,dt \geq 0 \text{ for } x \in \langle a+\lambda, b] , \tag{3.3}$$

in case $0 < \lambda \leq b-a$ and to

$$\int_a^x u(t)\,dt \geq \int_a^x w(t)\,dt \text{ for } x \in [a, b] \text{ and } \int_x^{a+\lambda} u(t)\,dt \leq 0 \text{ for } x \in \langle b, a+\lambda] ,$$

in case $\lambda > b-a$.

Further for $u \equiv 1$ we have $\int_a^b w(t)\,dt = \int_a^{a+\lambda} u(t)\,dt = \lambda$. Thus if $0 \leq w(t) \leq 1$ for $t \in [a,b]$ then $\lambda \leq b-a$ and it's easy to see that (3.3) is fulfilled. In a such a way the right-hand side of the Steffensen's inequality (1.1) is recaptured.

Corollary 4.2. *Let $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ be n-convex function on $[a,b] \cup [b-\lambda, b]$, $n \in \mathbb{N}$. Let $w_i : [a,b] \to [0,\infty\rangle$, $i = 1,..,n$ and $u_i : [b-\lambda, b] \to [0,\infty\rangle$, $i = 1,..,n$ be two sequences of weight functions as in Theorem 3. If*

$$K \leq 0 \tag{3.4}$$

where K is the function defined by (2.2), then for any $x \in [a,b] \cap [b-\lambda, b]$ it holds:

$$\begin{aligned} &\frac{1}{\int_a^b w_1(t)dt} \int_a^b w_1(t) f(t)\,dt + T_{w_1,..,w_n}^{[a,b]}(x) \geq \\ &\geq \frac{1}{\int_{b-\lambda}^b u_1(t)dt} \int_{b-\lambda}^b u_1(t) f(t)\,dt + T_{u_1,..,u_n}^{[b-\lambda,b]}(x) \end{aligned} \tag{3.5}$$

In case f is n-concave function inequality (3.5) holds if $K \geq 0$.

Proof. We apply Theorem 4 with $[c,d] = [b-\lambda, b]$. □

Remark 3.2: For every differentiable, nonincreasing function $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ and $w : [a,b] \to [0,\infty\rangle$ and $u : [b-\lambda, b] \to [0,\infty\rangle$ some weight functions such that $\int_a^b w(t)\,dt = \int_{b-\lambda}^b u(t)\,dt$ inequality (3.5) for $n = 1$ reduces to

$$\int_a^b w(t) f(t)\,dt \geq \int_{b-\lambda}^b u(t) f(t)\,dt$$

while condition $K \geq 0$ reduces to

$$\int_a^x w(t)\,dt \geq 0 \text{ for } x \in [a, b-\lambda] \text{ and } \int_{b-\lambda}^x u(t)\,dt \leq \int_a^x w(t)\,dt \text{ for } x \in \langle b-\lambda, b] \tag{3.6}$$

in case $0 < \lambda \leq b-a$ and to

$$\int_{b-\lambda}^x u(t)\,dt \leq 0 \text{ for } x \in [b-\lambda, a] \text{ and } \int_{b-\lambda}^x u(t)\,dt \leq \int_a^x w(t)\,dt \text{ for } x \in \langle a, b] ,$$

in case $\lambda > b-a$.

Further for $u \equiv 1$ we have $\int_a^b w(t)\,dt = \int_{b-\lambda}^b u(t)\,dt = \lambda$. Thus if $0 \leq w(t) \leq 1$ for $t \in [a,b]$ then $\lambda \leq b-a$ and it's easy to see that (3.6) is fulfilled since

$$x - b + \lambda = \int_{b-\lambda}^x u(t)\,dt \leq \int_a^x w(t)\,dt = \lambda - \int_x^b w(t)\,dt.$$

In a such a way the left-hand side of the Steffensen's inequality (1.1) is recaptured.

4. L_p INEQUALITIES

Here, the symbol $L^p_{[a,b]}$ $(1 \le p < \infty)$ denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_{p,[a,b]} = \left(\int_a^b |f(t)|^p \, dt\right)^{\frac{1}{p}}$$

and $L^\infty_{[a,b]}$denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_{\infty,[a,b]} = ess \sup_{t\in[a,b]} |f(t)| .$$

Theorem 5. *Suppose that all the assumptions of Theorem 3 hold. Additionally assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$, and $f^{(n)} \in L^p_{[a,b]\cup[c,d]}$. Then the following inequality holds*

$$\begin{aligned}
&\left|\frac{1}{\int_c^d u_1(t)\,dt}\int_c^d u_1(t) f(t)\,dt - T^{[a,b]}_{w_1,..,w_n}(x)\right.\\
&\left.-\frac{1}{\int_a^b w_1(t)\,dt}\int_a^b w_1(t) f(t)\,dt + T^{[c,d]}_{u_1,..,u_n}(x)\right|\\
&\le \|K(x,t_1,\dots,t_{n-1},\cdot)\|_{q,[\min\{a,c\},\max\{b,d\}]} \left\|f^{(n)}\right\|_{p,[\min\{a,c\},\max\{b,d\}]} \qquad (4.1)
\end{aligned}$$

Inequality (4.1) is sharp for $1 < p \le \infty$ and for $p = 1$ constant $\|K(x,t_1,\dots,t_{n-1},\cdot)\|_{q,[\min\{a,c\},\max\{b,d\}]}$ is the best possible.

Proof. By taking the modulus on (2.1) and applying the Hölder inequality we obtain

$$\begin{aligned}
&\left|\frac{1}{\int_c^d u_1(t)\,dt}\int_c^d u_1(t) f(t)\,dt - T^{[a,b]}_{w_1,..,w_n}(x)\right.\\
&\left.-\frac{1}{\int_a^b w_1(t)\,dt}\int_a^b w_1(t) f(t)\,dt + T^{[c,d]}_{u_1,..,u_n}(x)\right|\\
&= \left|\int_{\min\{a,c\}}^{\max\{b,d\}} K(x,t_1,\dots,t_n) f^{(n)}(t_n)\,dt_n\right|\\
&\le \|K(x,t_1,\dots,t_{n-1},\cdot)\|_{q,[\min\{a,c\},\max\{b,d\}]} \left\|f^{(n)}\right\|_{p,[\min\{a,c\},\max\{b,d\}]}
\end{aligned}$$

Let's denote $C(t) = K(x,t_1,\dots,t_{n-1},t)$. For the proof of the sharpness we will find a function f for which the equality in (4.1) is obtained.

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = sgn\ C(t) \cdot |C(t)|^{\frac{1}{p-1}} .$$

For $p=\infty$ take

$$f^{(n)}(t)=sgn\ C(t).$$

For $p=1$ we shall prove that

$$\left|\int_{\min\{a,c\}}^{\max\{b,d\}} C(t)\, f^{(n)}(t)\, dt\right| \leq \max_{t\in[\min\{a,c\},\max\{b,d\}]} |C(t)| \left(\int_{\min\{a,c\}}^{\max\{b,d\}} \left|f^{(n)}(t)\right| dt\right) \tag{4.2}$$

is the best possible inequality.

If $n\geq 2$ function $C(t)$ is continuous except in points $\max\{a,c\}$ and $\min\{b,d\}$ where it has a finite jump. If $n=1$ it is continuous. Thus we have four possibilities:

1. $|C(t)|$ attains its maximum at $t_0\in[\min\{a,c\},\max\{b,d\}]$ and $C(t_0)>0$. Then for $\varepsilon>0$ small enough define $f_\varepsilon(t)$ by

$$f_\varepsilon(t)=\begin{cases} 0, & \min\{a,c\}\leq t\leq t_0-\varepsilon, \\ \frac{1}{\varepsilon n!}(t-t_0+\varepsilon)^n, & t_0-\varepsilon\leq t\leq t_0, \\ \frac{1}{n!}(t-t_0+\varepsilon)^{n-1}, & t_0\leq t\leq \max\{b,d\}. \end{cases}$$

Thus

$$\left|\int_{\min\{a,c\}}^{\max\{b,d\}} C(t) f_\varepsilon^{(n)}(t)dt\right| = \left|\int_{t_0-\varepsilon}^{t_0} C(t)\frac{1}{\varepsilon}dt\right| = \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} C(t)dt.$$

Now, from inequality (4.2) we have

$$\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} C(t)dt \leq \frac{1}{\varepsilon}C(t_0)\int_{t_0-\varepsilon}^{t_0} dt = C(t_0).$$

Since

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} C(t)dt = C(t_0)$$

the statement follows.

2. $|C(t)|$ attains its maximum at $t_0\in[\min\{a,c\},\max\{b,d\}]$ and $C(t_0)<0$. Then for $\varepsilon>0$ small enough define $f_\varepsilon(t)$ by

$$f_\varepsilon(t)=\begin{cases} \frac{1}{n!}(t_0-t)^{n-1}, & \min\{a,c\}\leq t\leq t_0-\varepsilon, \\ -\frac{1}{\varepsilon n!}(t_0-t)^n, & t_0-\varepsilon\leq t\leq t_0, \\ 0, & t_0\leq t\leq \max\{b,d\}, \end{cases}$$

and the rest of proof is similar as above.

3. $|C(t)|$ does not attains a maximum on the $[\min\{a,c\},\max\{b,d\}]$ and let $t_0\in[\min\{a,c\},\max\{b,d\}]$ be such that

$$\sup_{t\in[\min\{a,c\},\max\{b,d\}]} |C(t)| = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} |f(t_0+\varepsilon)|$$

If $\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} f(t_0+\varepsilon)>0$, we take

$$f_\varepsilon(t)=\begin{cases} 0, & \min\{a,c\}\leq t\leq t_0, \\ \frac{1}{\varepsilon n!}(t-t_0)^n, & t_0\leq t\leq t_0+\varepsilon, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\varepsilon\leq t\leq \max\{b,d\}, \end{cases}$$

and similar as before we have

$$\left|\int_{\min\{a,c\}}^{\max\{b,d\}} C(t) f_{\varepsilon}^{(n)}(t) dt\right| = \left|\int_{t_0}^{t_0+\varepsilon} C(t) \frac{1}{\varepsilon} dt\right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C(t) dt,$$

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C(t) dt \leq \frac{1}{\varepsilon} C(t_0) \int_{t_0}^{t_0+\varepsilon} dt = C(t_0),$$

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} C(t) dt = C(t_0)$$

and the statement follows.

4. $|C(t)|$ does not attains a maximum on the $[\min\{a,c\}, \max\{b,d\}]$ and let $t_0 \in [\min\{a,c\}, \max\{b,d\}]$ be such that

$$\sup_{t \in [\min\{a,c\}, \max\{b,d\}]} |C(t)| = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} |f(t_0 + \varepsilon)|.$$

If $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(t_0 + \varepsilon) < 0$, we take

$$f_{\varepsilon}(t) = \begin{cases} \frac{1}{n!} (t - t_0 - \varepsilon)^{n-1}, & \min\{a,c\} \leq t \leq t_0, \\ -\frac{1}{\varepsilon n!} (t - t_0 - \varepsilon)^{n}, & t_0 \leq t \leq t_0 + \varepsilon, \\ 0, & t_0 + \varepsilon \leq t \leq \max\{b,d\}, \end{cases}$$

and the rest of proof is similar as above. □

Corollary 5.1. *Let $f : [a,b] \cup [a, a+\lambda] \to \mathbb{R}$ be such $f' \in L^p_{[a,b]\cup[a,a+\lambda]}$ and $g : [a,b] \to \mathbb{R}$ integrable function such $\lambda = \int_a^b g(t) dt$. Let also $G(x) = \int_a^x g(t) dt$, $x \in [a,b]$. Then the following two sharp inequalities hold for $1 < p \leq \infty$ and for $0 \leq \lambda \leq b - a$*

$$\left|\int_a^b f(t) g(t) dt - \int_a^{a+\lambda} f(t) dt\right|$$
$$\leq \left(\int_a^{a+\lambda} |t - a - G(t)|^q dt + \int_{a+\lambda}^b |\lambda - G(t)|^q dt\right)^{\frac{1}{q}} \left\|f'\right\|_{p,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b - a$

$$\left|\int_a^b f(t) g(t) dt - \int_a^{a+\lambda} f(t) dt\right|$$
$$\leq \left(\int_a^b |t - a - G(t)|^q dt + \int_b^{a+\lambda} |t - a - \lambda|^q dt\right)^{\frac{1}{q}} \left\|f'\right\|_{p,[a,\max\{b,a+\lambda\}]}.$$

In case $p = 1$ and $0 \leq \lambda \leq b - a$ we have following two best possible inequalities

$$\left|\int_a^b f(t) g(t) dt - \int_a^{a+\lambda} f(t) dt\right|$$

$$\leq \max\left\{\max_{t\in[a,a+\lambda]}\left|t-a-G(t)\right|,\max_{t\in[a+\lambda,b]}\left|\lambda-G(t)\right|\right\}\left\|f^{'}\right\|_{1,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b-a$

$$\left|\int_a^b f(t)\,g(t)\,dt-\int_a^{a+\lambda}f(t)\,dt\right|$$
$$\leq \max\left\{\max_{t\in[a,b]}\left|t-a-G(t)\right|,\max_{t\in[b,a+\lambda]}\left|t-a-\lambda\right|\right\}\left\|f^{'}\right\|_{1,[a,\max\{b,a+\lambda\}]}.$$

Proof. Applying Theorem 5 with $n=1$ and weight functions $w_1(t)=g(t)$ for $t\in[a,b]$ and $u_1(t)=1$ for $t\in[a,a+\lambda]$. We have $\int_a^b g(t)\,dt=\int_a^{a+\lambda}dt=\lambda$ and consequently

$$\left|\int_a^b f(t)\,g(t)\,dt-\int_a^{a+\lambda}f(t)\,dt\right|=\left|\lambda\int_a^{\max\{b,a+\lambda\}}K(t)\,f'(t)\,dt\right|$$

where

$$\lambda K(t)=\begin{cases} t-a-\int_a^t g(s)\,ds, & t\in[a,a+\lambda],\\ \int_t^b g(s)\,ds, & t\in\langle a+\lambda,b], \end{cases}\quad \text{if } a+\lambda\leq b,$$

$$\lambda K(t)=\begin{cases} t-a-\int_a^t g(s)\,ds, & t\in[a,b],\\ t-a-\lambda, & t\in\langle b,a+\lambda], \end{cases}\quad \text{if } a+\lambda> b,$$

and the proof follows. □

Corollary 5.2. *Let* $f:[a,b]\cup[b-\lambda,b]\to\mathbb{R}$ *be such* $f'\in L^p_{[a,b]\cup[b-\lambda,b]}$ *and* $g:[a,b]\to\mathbb{R}$ *integrable function such* $\lambda=\int_a^b g(t)\,dt$. *Let also* $G(x)=\int_a^x g(t)\,dt$, $x\in[a,b]$. *Then the following two sharp inequalities hold for* $1<p\leq\infty$ *and for* $0\leq\lambda\leq b-a$

$$\left|\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right|$$
$$\leq \left(\int_a^{b-\lambda} |-G(t)|^q\,dt + \int_{b-\lambda}^b |t-b+\lambda-G(t)|^q\,dt\right)^{\frac{1}{q}} \left\|f^{'}\right\|_{p,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b-a$

$$\left|\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right|$$
$$\leq \left(\int_{b-\lambda}^a |t-b+\lambda|^q\,dt + \int_a^b |t-b+\lambda-G(t)|^q\,dt\right)^{\frac{1}{q}} \left\|f^{'}\right\|_{p,[a,\max\{b,a+\lambda\}]}.$$

In case $p=1$ *and* $0 \leq \lambda \leq b-a$ *we have following two best possible inequalities*

$$\left|\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right|$$
$$\leq \max\left\{\max_{t\in[a,b-\lambda]} |-G(t)|,\ \max_{t\in[b-\lambda,b]} |t-b+\lambda-G(t)|\right\} \left\|f^{'}\right\|_{1,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b-a$

$$\left|\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right|$$
$$\leq \max\left\{\max_{t\in[b-\lambda,a]} |t-b+\lambda|,\ \max_{t\in[a,b]} |t-b+\lambda-G(t)|\right\} \left\|f^{'}\right\|_{1,[a,\max\{b,a+\lambda\}]}.$$

Proof. Applying Theorem 5 with $n=1$ and weight functions $w_1(t) = g(t)$ for $t \in [a,b]$ and $u_1(t) = 1$ for $t \in [b-\lambda, b]$. We have $\int_a^b g(t)\,dt = \int_{b-\lambda}^b dt = \lambda$ and consequently

$$\left|\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right| = \left|\lambda \int_{\min\{a,b-\lambda\}}^b K(t)\,f'(t)\,dt\right|$$

where

$$\lambda K(t) = \begin{cases} -G(t), & t \in [a, b-\lambda], \\ t-b+\lambda-G(t), & t \in \langle b-\lambda, b], \end{cases} \quad \text{if } a+\lambda \leq b,$$

$$\lambda K(t) = \begin{cases} t-b+\lambda, & t \in [b-\lambda, a], \\ t-b+\lambda-G(t), & t \in \langle a, b], \end{cases} \quad \text{if } a+\lambda > b,$$

and the proof follows. □

5. $k-$exponential convexity of Steffensen's inequality via n weight functions

Motivated by inequalities (2.6), (3.2), (3.5), and under assumptions of Theorem 4 and Corollaries 4.1 and 4.2, respectively, we define following linear functionals:

$$L_1(f) = \frac{1}{\int_c^d u_1(t)\,dt} \int_c^d u_1(t) f(t)\,dt + T_{u_1,..,u_n}^{[c,d]}(x) - \frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt - T_{w_1,..,w_n}^{[a,b]}(x) \tag{5.1}$$

$$L_2(f) = \frac{1}{\int_a^{a+\lambda} u_1(t)\,dt} \int_a^{a+\lambda} u_1(t) f(t)\,dt + T_{u_1,..,u_n}^{[a,a+\lambda]}(x) - \frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt - T_{w_1,..,w_n}^{[a,b]}(x) \tag{5.2}$$

$$L_3(f) = \frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt + T_{w_1,..,w_n}^{[a,b]}(x) - \frac{1}{\int_{b-\lambda}^b u_1(t)\,dt} \int_{b-\lambda}^b u_1(t) f(t)\,dt - T_{u_1,..,u_n}^{[b-\lambda,b]}(x) \tag{5.3}$$

Remark 5.1: Under the assumptions of Theorem 4 and Corollaries 4.1 and 4.2 respectively, it holds $L_i(f) \geq 0$, $i = 1, 2, 3$ for all n-convex functions f.

Also, we define $I_1 = [a, b] \cup [c, d]$, $I_2 = [a, b] \cup [a, a + \lambda]$, $I_3 = [a, b] \cup [b - \lambda, b]$, $\tilde{I}_1 = [a, b] \cap [c, d]$, $\tilde{I}_2 = [a, b] \cap [a, a + \lambda]$ and $\tilde{I}_3 = [a, b] \cup [b - \lambda, b]$. Now, we give mean value theorems for defined functionals.

Theorem 6. *Let $f : I_i \to \mathbb{R}$ $(i = 1, 2, 3)$ be such that $f \in C^n(I_i)$. If for $x \in \tilde{I}_i$ inequalities in (2.5) $(i = 1)$, (3.1) $(i = 2)$ and (3.4) $(i = 3)$ hold, then there exist $\xi_i \in \tilde{I}_i$ such that*

$$L_i(f) = f^{(n)}(\xi_i) L_i(\varphi), \quad i = 1, 2, 3 \tag{5.4}$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Let us denote $m = \min f^{(n)}$ and $M = \max f^{(n)}$. We consider the following functions $F_1(x) = \frac{Mx^n}{n!} - f(x)$ and $F_2(x) = f(x) - \frac{mx^n}{n!}$. Then $F_1^{(n)}(x) = M - f^{(n)} \geq 0$ and $F_2^{(n)}(x) = f^{(n)}(x) - m \geq 0$, for $x \in \tilde{I}_i$, so F_1 and F_2 are $n-$convex functions. Now we use inequalities from Theorem 4 and Corollaries 4.1 and 4.2 for $n-$convex functions F_1 i F_2, so we can conclude that there exists $\xi_i \in \tilde{I}_i$, $i = 1, 2, 3$ that we are looking for in (5.4). □

Theorem 7. *Let $f, g : I_i \to \mathbb{R}$ $(i = 1, 2, 3)$ be such that $f, g \in C^n(I_i)$. If for $x \in \tilde{I}_i$ inequalities in (2.5) $(i = 1)$, (3.1) $(i = 2)$ and (3.4) $(i = 3)$ hold, then there exist*

$\xi_i \in \tilde{I}_i$ such that

$$\frac{L_i(f)}{L_i(g)} = \frac{f^{(n)}(\xi_i)}{g^{(n)}(\xi_i)}, \quad i = 1, 2, 3. \tag{5.5}$$

assuming neither of the denominators is equal to zero.

Proof. For fix $1 \leq i \leq 3$ we define function $\Phi_i(x) = f(x)L_i(g) - g(x)L_i(f)$. According to Theorem 6 there exists $\xi_i \in \tilde{I}_i$ such that $L_i(\Phi_i) = \Phi_i^{(n)}(\xi_i)L_i(\varphi)$. Since $L_i(\Phi_i) = 0$ it follows that $f^{(n)}(\xi_i)L_i(g) - g^{(n)}(\xi_i)L_i(f) = 0$ and (5.5) is proved. □

We use previously defined functionals to construct exponentially convex functions, a special type of convex functions that are invented by S. N. Bernstein over eighty years ago in [8]. First, let us recall some definitions and facts about exponentially convex functions (see [13]).

Definition 5.1. *A function $\psi : I \to \mathbb{R}$ is k-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{k} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

holds for all choices $\xi_1, \ldots, \xi_k \in \mathbb{R}$ and all choices $x_1, \ldots, x_k \in I$. A function $\psi : I \to \mathbb{R}$ is k-exponentially convex if it is k-exponentially convex in the Jensen sense and continuous on I.

Remark 5.2: It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, k-exponentially convex function in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}$, $m \leq k$.

Definition 5.2. *A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is k-exponentially convex in the Jensen sense for any $k \in \mathbb{N}$.*

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 5.3: A positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex (see [9]).

Proposition 5.1. *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 5.3. *Let f be a real-valued function defined on the segment $[a, b]$. The divided difference of order n of the function f at distinct points $x_0, ..., x_n \in [a, b]$, is defined recursively (see [7], [14]) by*

$$f[x_i] = f(x_i), \ (i = 0, \ldots, n)$$

and

$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$.
The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\dots,x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{5.6}$$

Now, we use an idea from [9] to generate k-exponentially and exponentially convex functions applying defined functionals. In the sequel the notion log denotes the natural logarithm function.

Theorem 8. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1,2,3$ subset of $\mathbb{R}$ such that the function $p \mapsto f_p[x_0,\dots,x_n]$ is $k-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n \in I_i$, $i = 1,2,3$. Let L_i, $i = 1,2,3$ be linear functionals defined by (5.1)-(5.3). Then $p \mapsto L_i(f_p)$ is $k-$exponentially convex function in the Jensen sense on J.*
If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is $k-$exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R}$, $j = 1,\dots,k$ and $p_j \in J$, $j = 1,\dots,k$, we define the function

$$g(x) = \sum_{j,m=1}^{k} \xi_j \xi_m f_{\frac{p_j+p_m}{2}}(x).$$

Using the assumption that the function $p \mapsto f_p[x_0,\dots,x_n]$ is k-exponentially convex in the Jensen sense, we have

$$g[x_0,\dots,x_n] = \sum_{j,m=1}^{k} \xi_j \xi_m f_{\frac{p_j+p_m}{2}}[x_0,\dots,x_n] \geq 0,$$

which in turn implies that g is a n-convex function on J, so it is $L_i(g) \geq 0$, $i = 1,2,3$. Hence

$$\sum_{j,m=1}^{k} \xi_j \xi_m L_i\left(f_{\frac{p_j+p_m}{2}}\right) \geq 0.$$

We conclude that the function $p \mapsto L_i(f_p)$ is k-exponentially convex on J in the Jensen sense.

If the function $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is k-exponentially convex by definition. □

The following corollaries are the immediate consequences of the above theorem:

Corollary 8.1. *Let $\Omega = \{f_p : p \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$ subset of $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \dots, x_n]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I_i$. Let L_i, $i = 1, 2, 3$, be linear functionals defined as in (5.1)-(5.3). Then $p \mapsto L_i(f_p)$ is an exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

Corollary 8.2. *Let $\Omega = \{f_p : p \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$ subset of $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \dots, x_n]$ is 2-exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \dots, x_n \in I_i$. Let L_i, $i = 1, 2, 3$ be linear functionals defined as in (5.1)-(5.3). Then the following statements hold:*

(i) If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[L_i(f_s)]^{t-r} \le [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \le u$ and $q \le v$, we have

$$\mu_{p,q}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega), \tag{5.7}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \ne q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)}\right), & p = q, \end{cases} \tag{5.8}$$

for $f_p, f_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 8 and Remark 5.3.

(ii) Since the function $p \mapsto L_i(f_p)$, $i = 1, 2, 3$ is positive and continuous, according to (i) the function $p \mapsto L_i(f_p)$ is log-convex on J, and thus the function $p \mapsto \log L_i(f_p)$ is convex on J. Applying Proposition 5.1 we get

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p - q} \le \frac{\log L_i(f_u) - \log L_i(f_v)}{u - v}, \tag{5.9}$$

for $p \le u, q \le v, p \ne q, u \ne v$. Hence we conclude that

$$\mu_{p,q}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega).$$

Cases $p = q$ and $u = v$ follows from (5.9) as limit cases.

□

Remark 5.4: Note that the results from above theorem and corollaries still hold when two of the points $x_0, \dots, x_n \in I_i$, $i = 1, 2, 3$ coincide, say $x_1 = x_0$, for a family of differentiable functions f_p such that the function $p \mapsto f_p[x_0, \dots, x_n]$ is k-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all

$n+1$ points coincide for a family of n-times differentiable functions with the same property. The proofs use (5.6) and suitable characterization of convexity.

6. Applications to Stolarsky type means

In this section, we present several families of functions which fulfil the conditions of Theorem 8, Corollary 8.1, Corollary 8.2 and Remark 5.4. This enable us to construct a concrete examples of exponentially convex functions.

Example 6.1. *Consider a family of functions*

$$\Omega_1 = \{f_p : \mathbb{R} \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^n}, & p \neq 0, \\ \frac{x^n}{n!}, & p = 0. \end{cases}$$

We have $\frac{d^n f_p}{dx^n}(x) = e^{px} > 0$ *which shows that* f_p *is* n*-convex on* $\mathbb{R}$ *for every* $p \in \mathbb{R}$ *and* $p \mapsto \frac{d^n f_p}{dx^n}(x)$ *is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 8 we also have that* $p \mapsto f_p[x_0, \ldots, x_n]$ *is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 8.1 we conclude that* $p \mapsto L_i(f_p), i = 1, 2, 3,$ *are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping* $p \mapsto f_p$ *is not continuous for* $p = 0$*), so it is exponentially convex.*

For this family of functions, $\mu_{p,q}(L_i, \Omega_1),\ i = 1, 2, 3,$ *from (5.8), becomes*

$$\mu_{p,q}(L_i, \Omega_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{L_i(id \cdot f_p)}{L_i(f_p)} - \frac{n}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{1}{n+1} \frac{L_i(id \cdot f_0)}{L_i(f_0)}\right), & p = q = 0. \end{cases}$$

where id is the identity function. Also, by Corollary 8.2 it is monotonic function in parameters p *and* q*.*

We observe here that $\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}} (\log x) = x$ *so using Theorem 7 it follows that:*

$$M_{p,q}(L_i, \Omega_1) = \log \mu_{p,q}(L_i, \Omega_1),\ i = 1, 2, 3$$

satisfies

$$\min\{a, c, b - \lambda\} \leq M_{p,q}(L_i, \Omega_1) \leq \max\{a + \lambda, b, d\},\ i = 1, 2, 3.$$

So, $M_{p,q}(L_i, \Omega_1),\ i = 1, 2, 3$ *is monotonic mean.*

Example 6.2. *Consider a family of functions*

$$\Omega_2 = \{g_p : (0, \infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \frac{x^p}{p(p-1)\cdots(p-n+1)}, & p \notin \{0, 1, \ldots, n-1\}, \\ \frac{x^j \ln x}{(-1)^{n-1-j} j!(n-1-j)!}, & p = j \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Here, $\frac{d^n g_p}{dx^n}(x) = x^{p-n} = e^{(p-n)\ln x} > 0$ *which shows that* g_p *is* n*-convex for* $x > 0$ *and* $p \mapsto \frac{d^n g_p}{dx^n}(x)$ *is exponentially convex by definition. Arguing as in Example 6.1 we get that the mappings* $p \mapsto L_i(g_p), i = 1,2,3$ *are exponentially convex. In this case we assume that* $I_i \in \mathbb{R}^+$. *Functions (5.8) is now equal to:*

$$\mu_{p,q}(L_i,\Omega_2) = \begin{cases} \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, p \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\,\frac{L_i(g_0 g_p)}{L_i(g_p)} + \sum_{k=0}^{n-1}\frac{1}{k-p}\right), \\ \qquad\qquad p = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\,\frac{L_i(g_0 g_p)}{2L_i(g_p)} + \sum_{\substack{k=0\\k\neq p}}^{n-1}\frac{1}{k-p}\right), \\ \qquad\qquad p = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 7 we conclude that

$$\min\{a,c,b-\lambda\} \le \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}} \le \max\{b,d,a+\lambda\},\ i = 1,2,3,$$

which shows that $\mu_{p,q}(L_i,\Omega_2), i = 1,2,3$ *is mean.*

Example 6.3. *Consider a family of functions*

$$\Omega_3 = \{\phi_p : (0,\infty) \to \mathbb{R} : p \in (0,\infty)\}$$

defined by

$$\phi_p(x) = \begin{cases} \frac{p^{-x}}{(-\ln p)^n}, & p \neq 1 \\ \frac{x^n}{n!}, & p = 1. \end{cases}$$

Since $\frac{d^n \phi_p}{dx^n}(x) = p^{-x}$ *is the Laplace transform of a non-negative function (see* [16]*) it is exponentially convex. Obviously* ϕ_p *are* n*-convex functions for every* $p > 0$. *For this family of functions,* $\mu_{p,q}(L_i,\Omega_3), i = 1,2,3,$ *in this case for* $I_i \in \mathbb{R}^+$, *from (5.8) becomes*

$$\mu_{p,q}(L_i,\Omega_3) = \begin{cases} \left(\frac{L_i(\phi_p)}{L_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{L_i(id\cdot\phi_p)}{p_i(\phi_p)} - \frac{n}{p\ln p}\right), & p = q \neq 1, \\ \exp\left(-\frac{1}{n+1}\frac{L_i(id\cdot\phi_1)}{L_i(\phi_1)}\right), & p = q = 1. \end{cases}$$

This is monotone function in parameters p *and* q *by (5.7). Using Theorem 7 it follows that*

$$M_{p,q}(L_i,\Omega_3) = -L(p,q)\log\mu_{p,q}(L_i,\Omega_3),\ i = 1,2,3$$

satisfy

$$\min\{a,c,b-\lambda\} \le M_{p,q}(L_i,\Omega_3) \le \max\{b,d,a+\lambda\},\ i = 1,2,3.$$

So $M_{p,q}(L_i,\Omega_3)$ *is monotonic mean.* $L(p,q)$ *is logarithmic mean defined by*

$$L(p,q) = \begin{cases} \frac{p-q}{\log p - \log q}, & p \neq q \\ p, & p = q. \end{cases}$$

Example 6.4. *Consider a family of functions*

$$\Omega_4 = \{\psi_p : (0,\infty) \to \mathbb{R} : p \in (0,\infty)\}$$

defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{(-\sqrt{p})^n}.$$

Since $\frac{d^n\psi_p}{dx^n}(x) = e^{-x\sqrt{p}}$ *is the Laplace transform of a non-negative function (see* [16]*) it is exponentially convex. Obviously* ψ_p *are* n*-convex functions for every* $p > 0$*. For this family of functions,* $\mu_{p,q}(L_i,\Omega_4), i = 1,2,3$ *from* (5.8) *is equal to*

$$\mu_{p,q}(L_i,\Omega_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{L_i(id\cdot\psi_p)}{2\sqrt{p}L_i(\psi_p)} - \frac{n}{2p}\right), & p = q, \end{cases}$$

where id *is the identity function. This is monotone function in parameters* p *and* q *by* (5.7)*. Using Theorem 7 it follows that*

$$M_{s,q}(L_i,\Omega_4) = -(\sqrt{p}+\sqrt{q})\log\mu_{p,q}(L_i,\Omega_4),\ \ i = 1,2,3$$

satisfies $\min\{a,c,b-\lambda\} \leq M_{p,q}(L_i,\Omega_4) \leq \max\{b,d,a+\lambda\}$*, so* $M_{p,q}(L_i,\Omega_4)$*,* $i = 1,2,3$ *is monotonic mean.*

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