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ON SOME INEQUALITIES FOR CONVEX FUNCTIONS AND SOME RELATED APPLICATIONS

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1. Let $x_i \in I$, and $p_i \geqslant 0$ $(i = 1, \ldots, n)$ are real numbers, f is convex function on I and

$$m = min x_i, M = max x_i, \bar{x} = \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}.$$

Then JENSEN and converse JENSEN inequalities hold (see [1]), i.e.

$$f(\bar{x}) \leqq \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} \leqq \frac{M-\bar{x}}{M-m} f(m) + \frac{\bar{x}-m}{M-m} f(M). \tag{1}$$

From (1), we can get the following result:

THEOREM 1. *Let $x_1, \ldots, x_n$ be equidistant points from I. Then for every convex function f on I*

$$f\left(\frac{x_1 + x_n}{2}\right) \leqq \frac{1}{n} \sum_{i=1}^{n} f(x_i) \leqq \frac{f(x_1) + f(x_n)}{2} \tag{2}$$

If $n > 2$ inequality is strict unless f is a linear function.

PROOF. If $p_1 = 1$ $(i = 1, \ldots, n)$, then $m = x_1, M = x_n$, $\bar{x} = (x_1 + x_n)/2$, and from (1) we obtain (2).

REMARK 1. Inequalities (2) are related to HADAMARD' s inequalities, which, analogously to the previous proof, we can get from integral analogue of (1).

COROLLARY 1. *The function* $f(x) = -\ln x$ *is convex for* $x>0$, *so, from* (2) *we get*

$$(3)\qquad (x_1 x_n)^{\frac{n}{2}} < \prod_{k=1}^{n} x_k < \left(\frac{x_1 + x_n}{2}\right)^n \quad (n > 2),$$

where $(x_i)_{i=1,\ldots,n}$ *is an arithmetical sequence.*

REMARK 2. The inequalities (3) are generalizations of some results from [2, pp. 190].

COROLLARY 2. *Let* $(x_i)_{i=1,\ldots,n}$ *and* $(y_i)_{i=1,\ldots,n}$ $(n > 2)$, *are two positive increasing arithmetical sequences and let*

$$(4)\qquad x_1 y_n > x_n y_1.$$

Then

$$(5)\qquad \left(\frac{x_1 + x_n}{y_1 + y_n}\right)^n < \prod_{i=1}^{n} \frac{x_i}{y_i} < \left(\frac{x_1 x_n}{y_1 y_n}\right)^{\frac{n}{2}}$$

If the reverse inequality in (4) holds then the reverse inequality in (5) holds, too.

PROOF. The function $f(x) = \ln \dfrac{ax - b}{cx - d}\left(a, b, c, d > 0,\ x > \max\left(\dfrac{b}{a}, \dfrac{d}{c}\right)\right)$ is convex for $ad > cb$ and concave for $ad < cb$. From the other point of view there is always existance of an increasing sequence $(\alpha_i)_{i=1,\ldots,n}$ such that $x_i = a\alpha_i - b$ and $y_i = c\alpha_i - d$, so from (2) we obtain (5).

REMARK 3. Using (5) (and (3)) we can easily get some inequalities for $\binom{r}{n}$ where $n \in \mathrm{N}$ and $r \in N$ or R. For example

$$\left(\frac{2r - n + 1}{n + 1}\right)^n < \binom{r}{n} < \frac{1}{n!}\left(\frac{2r - n + 1}{2}\right)^n \quad (n > 2).$$

wherefrom an inequality from [3] follows (see also [2, pp. 192]).

COROLLARY 3. *Let* $(x_i)_{i=1,\ldots,n}$ *be positive increasing arithmetical sequence. Then*

$$(6)\qquad \Gamma\left(\frac{x_1 + x_n}{2}\right)^n = \prod_{i=1}^{n} \Gamma(x_i) = (\Gamma(x_1)\,\Gamma(x_n))^{n/2}$$

PROOF. The function $x \to \ln \Gamma(x)$ is convex, so from (2) we obtain (6).

REMARK 4. The inequalities (6) are generalizations of some results from [2, p.].

COROLLARY 4. *The function $f(x) = x\ln h$ is convex for $x\,0$, so from* (2) *we get*

$$\left(\frac{x_1 + x_n}{2}\right)^{\frac{x_1 + x_2}{2}} < \sqrt[n]{x_1^{x_1} \cdots x_n^{x_n}} < \sqrt{x_1^{x_1}\, x_n^{x_n}} \quad (n > 2)$$

2. **THEOREM** 2. *Let $(x_i)_{i}=0,1\,..,n+1$ is an increasing arithmetical sequences, $x_{k+1} - x_k = h$ and $x_i \in I$ $(i=0, 1, \ldots, n+1)$. Then for every conven and differentiable function f on I the following hold*:

$$(7)\qquad \frac{f(x_n) - f(x_o)}{h} < \frac{f(x_n) - f(x_1)}{h} + f'(x_1) < \sum_{k=1}^{n} f'(x_k)$$

$$< \frac{f(x_n) - f(x_1)}{h} + f'(x_n) < \frac{f(x_{n+1}) - f(x_1)}{h}.$$

If f is a concave function then the reverse inequalities are valid.

PROOF. Let f be a convex and differentiable function on I. Then (see for example [4]):

$$(8)\qquad f(x) + hf'(x) < f(x+h) < f(x) + hf'(x+h) \qquad (h \neq 0),$$

where $x,\ h \in R$ and $x, x+h \in I$.

For concave function signs of inequality change their sense. From (8) we get

$$(9)\qquad f(x_k) + hf'(x_k) < f(x_{k+1}) < f(x_k) + hf'(x_{k+1}),$$

Summing these series of inequalities for k $= 1, \ldots, n-1$, we obtain

$$\sum_{k=1}^{n-1} f(x_k) + h\sum_{k=1}^{n-1} f'(x_k) < \sum_{k=2}^{n} f(x_k) < \sum_{k=1}^{n-1} f(x_k) + h\sum_{k=2}^{n} f'(x_k),$$

wherefrom we get the second and the third inequality from (7). The first and the fourth inequality from (7) we obtain on basis (9).

COROLLARY 4. *For $f(x) = x^{m+1}$, from* (7), *we get for each $m > 0$*

$$(10)\qquad \frac{x_n^{m+1} - x_o^{m+1}}{h(m+1)} < \frac{x_n^{m+1} - x_1^{m+1}}{h(m+1)} + x_1^m < \sum_{k=1}^{n} x_k^m < \frac{x_n^{m+1} - x_1^{m+1}}{h(m+1)} +$$

$$+ x_n^m < \frac{x_{n+1}^{m+1} - x_1^{m+1}}{h(m+1)}.$$

If $m < 0$ $(m \neq -1)$ *the reverse inequalities are valid.*

From the second and the third inequalities we get

$$\lim_{n\to\infty} \frac{\sum_{k=1} x_k^m}{x_n^{m+1}} = \frac{1}{h(m+1)} \quad (m > -1)$$

the generalization of the result from [5].

REMARK 5. For $x_i = i\,(i = 1, \ldots, n)$, from (10), we obtain the result from [4]

COROLLARY 5. *For* $f(x) = \ln x$ *we obtain*

$$\frac{1}{h} \ln \frac{x_{n+1}}{x_1} < \frac{1}{h} \ln \frac{x_n}{x_1} + \frac{1}{x_n} < \sum_{k=1}^{n} \frac{1}{x_k} < \frac{1}{h} \ln \frac{x_n}{x_1} + \frac{1}{x_1} < \frac{1}{h} \ln \frac{x_n}{x_o}.$$

If $x_1 = n$, $h = 1$, we get

$$\ln \frac{s+1}{n} < \ln \frac{s}{n} + \frac{1}{s} < \sum_{k=n}^{s} \frac{1}{k} < \ln \frac{s}{n} + \frac{1}{n} < \ln \frac{s}{n-1}. \tag{11}$$

Denote

$$s_n(p, q) = \sum_{k=pn+1}^{cn} \frac{1}{k}, \qquad s_n(p, q) = \sum_{k=pn}^{qn-1} \frac{1}{k} \quad (q, p \in N,\ q > p).$$

Then the following result holds:

COROLLARY 6. *For fixed natural numbers p and q* $(>p)$, *the sequence* $s_n(p, q)$ *is strictly increasing, and the sequence* $S_n(p, q)$ *is strictly decreasing and*

$$\begin{aligned} \ln \frac{q+1}{p+1} < \ln \frac{q}{p+1} + \frac{1}{q} < s_1(p, q) \leqq s_n(p, q) < \ln \frac{p}{q} < S_n(p, q) \\ \leqq S_1(p, q) < \ln \frac{q-1}{p} + \frac{1}{p} < \ln \frac{q-1}{p-1}. \end{aligned} \tag{12}$$

The bounds $s_1(p, q)$, $\ln \frac{p}{q}$, $S_1(p, q)$ *are the best possible for two fixed natural numbers p and q* $(> p)$.

PROOF. The assertion for $s_n(p, q)$ is proved in [6]. By analogy we can prove the assertion for $S_n(p, q)$. Namely, we have

$$S_n(p, q) = \sum_{s=p}^{q-1} S_n(s, s+1).$$

Hence, it suffices to prove that for $p=1, 2,..$ $S_n\ (p, p+1)$ increases, i. e.

$$\alpha_n(p) \overset{def}{=} S_{n+1}(p,p+1) - S_n(p,p+1) < 0 \quad (n,p=1,2,..).$$

Since

$$\frac{n+1}{n}\frac{1}{pn+p+1} - \frac{1}{pn+k} = \frac{k}{n(pn+p+k)(pn+k)} > 0.$$

or

$$\frac{1}{pn+p+k} > \frac{n}{n+1}\frac{1}{pn+1},$$

we get

$$\alpha_n(p) = \sum_{k=p(n+1)}^{(p+1)(n+1)-1} \frac{1}{k} - \sum_{k=pn}^{(p+1)n-1} \frac{1}{k}$$

$$= \frac{1}{pn+p} - \sum_{k=1}^{n}\left(\frac{1}{pn+k-1} - \frac{1}{pn+p+k}\right)$$

$$\leqq \frac{1}{pn+p} - \sum_{k=1}^{n} \frac{p+1}{(pn+k-1)(pn+p+k)}$$

$$\leqq \frac{1}{pn+p} - \frac{n}{n+1}\sum_{k=1}^{n} \frac{P+1}{(pn+k-1)(pn+k)} = 0$$

for $p, n = 1, 2, \ldots$ Using (11) we easily get (12).

Since, from (11)

$$\lim_{n\to\infty} s_n\ (p, q) = ln\frac{p}{q} \qquad \lim_{n\to\infty} S_n\ (p, q) = ln\ \frac{p}{q}$$

is valid, the bounds are the best possible.

REMARK 6. Corolary 6 is generalization of inequality from [7] (see also [2, pp. 186]).

3. The JENSEN-STEFFENSEN inequality ([8]) may be written in discrete form like this:

If f is a convex function on I and the sequence $x_k \in I\ (k=1,\ldots,n)$ is monotonous, and if the numbers $p_k\ (k=1,\ldots,n)$ satisfy the conditions

$$O \leqq P_k \leqq P_n \quad (k=1,..,n-1),\ P_n > O,\ \left(P_k = \sum_{i=1}^{k} p_i\right)$$

then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{13}$$

For $n = 2m-1, p_k = (-1)^{k-1}$ $(k = 1, \ldots, 2m-1)$ we get

$$(14) \qquad \sum_{i=1}^{2m-1} (-1)^{i-1} f(x_i) \geqq f\left(\sum_{i=1}^{2m-1} (-1)^{i-1} x_i.\right)$$

REMARK 7. The inequality (14) is proved in [9] for $x_1 > x_2 \ldots x_{2m-1} > 0$.

From (13), for $x_1 \geqq \ldots \geqq x_n \geqq 0$ and $f(0) \leqq 0$, we can get (see [10] or [11]):

$$(15) \qquad \sum_{k=1}^{n} (-1)^{k-1} f(x_k) \geqq f\left(\sum_{k=1}^{n} (-1)^{k-1} x_k).\right)$$

Now, we shall prove the following theorem:

THEOREM 3. *Let f be convex function on I $(0 \in I)$, $f(0) \leqq 0$, and*

$$(16) \qquad a_{i-1} \leqq b_{i-1} \leqq a_i \ (i = 2, \ldots, n),\ 0 \leqq b_n, \sum_{k=1}^{n} b_k \leqq \sum_{k=1}^{n} a_k.$$

Then

$$(17) \qquad \sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(b_i) \geqq f\left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right).$$

PROOF. Let $m = n$, $x_{2k-1} = a_k$, $x_{2k} = b_k$ $(k = 1, \ldots, n)$. The conditions for the application of (14) are fulfilled. So we have

$$\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n-1} f(b_i) \geqq f\left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i\right),$$

i. e.

$$(18) \qquad \sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(b_i) \geqq f\left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i\right) - f(b_n).$$

Let $n = 2, x_1 = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i$, $x_2 = b_n$. The conditions for the application of (15) are fulfilled. So we have

$$(19) \qquad f\left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} b_i\right) - f(b_n) \geqq f\left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right).$$

From (18) and (19) we get (17).

REMARK 8. Theorem 3, for nonnegative real numbers a_i and b_i $(i = 1, \ldots, n)$ and for $f(0) = 0$ is proved in [12] in defferent way.

COROLLARY 7. *If, in* (16_{*}), *is*

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$$

and if f is convex function on I $(0 \in I)$, *then*

$$\sum_{i=1}^{n} f(a_i) \geqq \sum_{i=1}^{n} f(b_i). \tag{20}$$

PROOF. Let f be a convex function on I $(0 \in I)$. Then function $x \to f(x) - f(0)$ satisfies the conditions of the Theorem 3. So from (17) we get (20).

COROLLARY 8. *Let* $n = 2$, $a_1 = a$, $a_2 = b + c$, $b_1 = a + b$, $b_2 = c$, *in Corollary* 7. *Then for* $0 < a \leqq c$, $0 < b$ *we obtain*

$$f(a) + f(b + c) \geqslant f(a + b) + f(c) \tag{21}$$

for every function f *convex for* $x \geqq 0$.

REMARK 9. Corollary 8 is given in [13] (see also [14]).

By analogy we can prove:

COROLLARY 9. *Let f be increasing convex function on* I $(0 \in I)$ *and let real numbers* a_i *and* b_i $(i = 1, \ldots, n)$ *satisfy the condition* (16). *Then the inequality*(20) *is valid.*

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