

ON TOPOLOGICAL n -GROUPS

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In this note it is shown that each topological n -group can be embedded into a topological group.

At first, some preliminary definitions and results will be stated.

Let Q be a non-empty set and:

$$\begin{aligned} (\dots) &: (x_0, \dots, x_n) \rightarrow (x_0 \dots x_n) \\ [.. \setminus] &: (x_0, \dots, x_n) \rightarrow [x_n \dots x_1 \setminus x_0] \\ [/\dots] &: (x_0, \dots, x_n) \rightarrow [x_0 / x_n \dots x_2] \end{aligned} \quad (1)$$

be three $n + 1$ -ary operations on $Q \cdot Q$ is said to be an n -group if the following identities are satisfied.

$$((x_0 \dots x_n) \dots x_{2n}) = (x_0 (x_1 \dots x_{n+1}) \dots x_{2n}) = \dots = (x_0 \dots (x_n \dots x_{2n})) \quad (2)$$

$$(x_1 \dots x_n [x_n \dots x_1 \setminus x]) = x = ([x / x_n \dots x_1] x_1 \dots x_n). \quad (3)$$

An n -group Q is said to be a topological n -group if Q is a topological space such that the operations (\dots) , $[.. \setminus]$ and $[/\dots]$ are continuous (in the all variables together).

The following result is known as Post's Coset Theorem¹⁾. If Q is an n -group then there is a unique (within to a canonical isomorphism) group G such that:

$$G = QUQ^2U \dots UQ^n \quad (4)$$

$$(\forall a_0, \dots, a_n \in Q) (a_0 \dots a_n) = a_0 \dots a_n \quad (5)$$

$$1 \leq i \leq j \leq n, a_\nu, b_\lambda \in Q \Rightarrow \{a_1 \dots a_i = b_1 \dots b_i \Leftrightarrow \quad (6)$$

$$\Leftrightarrow [i=j \ \& \ (\exists c_0, \dots, c_{n-i} \in Q) (c_0 \dots c_{n-i} a_i \dots a_i = (c_0 \dots c_{n-i} b_1 \dots b_i))\}.$$

1) [1] p. 37, or [5] p. 218

The group G is said to be the free covering of the given n -group Q .

Now we shall state and prove the following

Theorem. Let Q be a topological n -group, and G the free covering of Q . Define a collection \mathcal{B} of subsets of G by:

$$\mathcal{B} = \{A_1 \cdots A_k \mid 1 \leq k \leq n, A_1, \dots, A_k \text{ are open in } Q\} \quad (7)$$

Then we have:

- (i) \mathcal{B} is a base of a topology \mathcal{G} on G .
- (ii) The given topology on Q is induced by \mathcal{G} on Q , and Q is an open and closed subset of G .
- (iii) G is a topological group.
- (iv) If Q is a compact (Hausdorff) n -group then G is a compact (Hausdorff) group.

First, we prove three lemmas.

1. If $A_0, \dots, A_n \subseteq Q$, and A_n is an open subset of Q , then $(A_0 \cdots A_n) = A$ is an open subset of Q too.

Proof. If a_1, \dots, a_n are arbitrary elements of Q then the mappings: $f(x) = (a_1 \cdots a_n x)$ and $f^{-1}(x) = [a_n \cdots a_1 \setminus x]$ are continuous, and therefore $f(x)$ is a homeomorphism.

2. Let $a_1, \dots, a_i, b_1, \dots, b_i \in Q$, $1 \leq i \leq n$ and B_1, \dots, B_i be (open) neighborhoods of b_1, \dots, b_i . If $a_1 \cdots a_i = b_1 \cdots b_i$ (in G) then there exist neighborhoods A_1, \dots, A_i of a_1, \dots, a_i such that $A_1 \cdots A_i \subseteq B_1 \cdots B_i$.

Proof. By (6) there exist $c_0, \dots, c_{n-i} \in Q$ such that $c = (c_0 \cdots c_{n-i} a_1 \cdots a_i) = (c_0 \cdots c_{n-i} b_1 \cdots b_i)$, and (by 1) $U = (c_0 \cdots c_{n-i} B_1 \cdots B_i)$ is a neighborhood of c . Therefore there exist neighborhoods C_0, \dots, C_{n-i} of c_0, \dots, c_{n-i} , and A_1, \dots, A_i of a_1, \dots, a_i such that

$$(C_0 \cdots C_{n-i} A_1 \cdots A_i) \subseteq U = (c_0 \cdots c_{n-i} B_1 \cdots B_i)$$

e.

$$c_0 \cdots c_{n-i} A_1 \cdots A_i \subseteq C_0 \cdots C_{n-i} A_1 \cdots A_i \subseteq c_0 \cdots c_{n-i} B_1 \cdots B_i,$$

and thus we have $A_1 \cdots A_i \subseteq B_1 \cdots B_i$.

3. If $a_1, \dots, a_i, b_1, \dots, b_{n-i} \in Q$, and $s = a_1 \cdots a_i$ (in G), then

$$s^{-1} = [b/b_1 a_i \cdots a_1 b_{n-i} \cdots b_2] b_2 \cdots b_{n-i}. \quad (8)$$

Proof. By (3) and (5) we have $[x/x_n \cdots x_1] = x x_n^{-1} \cdots x_1^{-1}$, and this implies (8).

Proof of Theorem. (i) By (4) and (7), $G = \bigcup_{B \in \mathcal{B}} B$. Assume that $g \in$

$A_1 \dots A_i \cap B_1 \dots B_i$, where $1 \leq i \leq n$, and A_ν, B_ν are open in Q . Then there exist $a_\nu \in A_\nu, b_\nu \in B_\nu$, such that $g = a_1 \dots a_i = b_1 \dots b_i$. By 2, there exist neighborhoods A_1', \dots, A_i' of a_1, \dots, a_i such that $A_1' \dots A_i' \subseteq B_1 \dots B_i$, and thus $g \in A_1'' \dots A_i'' \subseteq A_1 \dots A_i \cap B_1 \dots B_i$, where $A_\nu'' = A_\nu \cap A_\nu'$. This proves that \mathcal{B} is a base of a topology \mathcal{T} on G .

(ii) By (6) and (7), Q, Q^2, \dots , and Q^n are disjoint open subsets of G , and therefore they are closed too.

If $A \subseteq Q$ and if A is an open set in the given topology on Q , then $A \in \mathcal{B}$, i. e. A is open in G too. Conversely, if A is an open subset of G and $A \subseteq Q$, then $A = \bigcup_{i \in I} B_i$, where $B_i \in \mathcal{B}$; by (6), for each $i \in I$, there exist open sets A_1, \dots, A_{k_i} ($1 \leq k_i \leq n$) such that $B_i = A_1 \dots A_{k_i}$, and this, implies that $k_i = 1$, i. e. $B_i = A_1$ is an open subset of Q , and therefore A is an open subset of Q too. This proves that the given topology on Q is induced by \mathcal{T} .

(iii) Let $s = a_1 \dots a_i, t = b_1 \dots b_j$ ($i, j \leq n, a_\nu, b_\lambda \in Q$) be two elements of G , and $g = st$. Let $C \in \mathcal{B}$ and $g \in C$. Then, if $C = C_1 \dots C_k$ ($k \leq n$), where C_1, \dots, C_k are open sets in Q , there exist $c_1, \dots, c_k \in Q$ such that $c_\nu \in C_\nu$, and $g = c_1 \dots c_k$. Thus:

$$a_1 \dots a_i b_1 \dots b_j = c_1 \dots c_k.$$

If $i + j \leq n$, then $i + j = k$, and by 1 there exist neighborhoods A_1, \dots, A_i of a_1, \dots, a_i and B_1, \dots, B_j of b_1, \dots, b_j such that

$$A_1 \dots A_i B_1 \dots B_j \subseteq C_1 \dots C_k = C, \quad (9)$$

where $A = A_1 \dots A_i$ is a neighborhood of s and $B = B_1 \dots B_j$ is a neighborhood of t .

If $i + j > n$ and if we put $a = (a_1 \dots a_i b_1 \dots b_{n-i+1})$, then we have $a b_{n-i+2} \dots b_j = c_1 \dots c_k$ and $k = i + j - n$; by 1 there exist neighborhoods A' of a and B_{n-i+2}, \dots, B_j of b_{n-i+2}, \dots, b_j such that $A' B_{n-i+2} \dots B_j \subseteq C$. From the equation $a = (a_1 \dots a_i b_1 \dots b_{n-i+1})$ follows that there exist neighborhoods A_1, \dots, A_i of a_1, \dots, a_i and B_1, \dots, B_{n-i+1} of b_1, \dots, b_{n-i+1} such that $(A_1 \dots A_i B_1 \dots B_{n-i+1}) \subseteq A'$, and this implies that $AB \subseteq C$, where $A = A_1 \dots A_i, B_1 \dots B_j = B$. This completes the proof that the operation " ." of the group G is continuous.

Assume now that $s = a_1 \dots a_i \in G$, where $1 \leq i \leq n, a_r \in Q$. If b_1, \dots, b_{n-i} are arbitrary elements of Q , then by 3 we have $s^{-1} = b b_2 \dots b_{n-1}$,

where $b = [b_1/b_1 a_i \cdots a_1 b_{n-i} \cdots b_2]$. If $C = B' C_2 \cdots C_{n-i}$ is a neighborhood of s^{-1} , then $s^{-1} = b' c_2 \cdots c_{n-i} = b b_2 \cdots b_{n-i}$, where $b' \in B'$, $c_v \in C_v'$. By 2 there exist neighborhoods B, B_2, \dots, B_{n-i} of b, b_2, \dots, b_{n-i} such that $BB_2 \cdots B_{n-i} \subseteq C$. From $b \in B$ and $b = [b_1/b_1 a_i \cdots a_1 b_{n-i} \cdots b_2]$ follows that there exist neighborhoods B_1', B_2'' of b_1, A_1, \dots, A_i of a_1, \dots, a_i and B_2', \dots, B'_{n-i} of b_2', \dots, b'_{n-i} , such that $[B_1'/B_1'' A_1 \cdots \cdots B_2'] \subseteq B$. Then we have:

$$\begin{aligned} (A_1 \cdots A_i)^{-1} &= [b_1/b_1 A_i \cdots A_1 b_{n-i} \cdots b_2] b_2 \cdots b'_{n-i} \subseteq \\ &\subseteq [B_1'/B_1'' A_i \cdots A_1 B_{n-i} \cdots B_2] B_2 \cdots B_{n-i} = C. \end{aligned}$$

Thus we have proved that the mapping $s \rightarrow s^{-1}$ is continuous, and this completes the proof of the statement (iii).

(iv) If the given space Q is compact then each cartesian product $Q \times Q \times \cdots \times Q$ is a compact space, and hence Q^k is a compact subset of G , because the mapping $(x_1, \dots, x_k) \rightarrow x_1 \cdots x_k$ is continuous. Then G is a compact space for it is a union of n compact subsets Q, Q^2, \dots, Q^n .

Assume now Q to be a Hausdorff space, and let $s = a_1 \cdots a_i$, $t = b_1 \cdots b_j$ ($a_v, b_\lambda \in Q$, $1 \leq i \leq j \leq n$) be two different elements of G . If $i \neq j$, then Q^i is a neighborhood of s , and Q^j is a neighborhood of t , where $Q^i \cap Q^j = \emptyset$. If $i = j$, then for arbitrary $c_0, \dots, c_{n-i} \in Q$ we have:

$$(c_0 \cdots c_{n-i} a_1 \cdots a_i) = a \neq b = (c_0 \cdots c_{n-i} b_1 \cdots b_i),$$

and therefore there are a neighborhood A of a and a neighborhood B of b such that $A \cap B = \emptyset$. Also, there exist neighborhoods C_0, \dots, C_{n-i} of c_0, \dots, c_{n-i} , A_1, \dots, A_i of a_1, \dots, a_i and B_1, \dots, B_i of b_1, \dots, b_i such that

$$(C_0 \cdots C_{n-i} A_1 A_i) \subseteq A, (C_0 \cdots C_{n-i} B_1 \cdots B_i) \subseteq B,$$

whence follows

$$c_0 \cdots c_{n-i} A_1 \cdots A_i \cap c_0 \cdots c_{n-i} B_1 \cdots B_i = \emptyset,$$

i. e. $A \cap B = \emptyset$, where $A = A_1 \cdots A_i$ is a neighborhood of s , and $B = B_1 \cdots B_i$ is a neighborhood of t . This completes the proof that G is a Hausdorff group.

Some remarks

a) It is well known ([5], p. 21) that the notions of T_0, T_1 and T_2 spaces are equivalent in the class of topological groups. Is the same statement true in the class of topological n -groups?

b) An algebra $Q(\dots)$ with an $n+1$ — ary operation (\dots) is said to be an n -semigroup if the identities (2) are satisfied. Then ([2], p. 23) there is a semigroup G (the free covering semigroup of Q) such that (4) and (5) are satisfied. If in addition the operation (\dots) is continuous (in the all variables together) then Q is said to be a topological n -semigroup. Are the statements of Theorem true in the class of topological n -semigroups?

It is known ([3]) that every topological universal algebra $A(\Omega)$ may be (in a corresponding way) embedded into a topological semigroup S , but even in the case when A is a topological n -semigroups, S is not the free covering of A .

c) A group G is said to be semitopological if G is a topological space in which each translation (left or right) is continuous. This notion for n - groups. can be generalized in an obvious way. Are the statements of Theorem true for semitopological n -groups?

R E F E R E N C E S

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ЗА ТОПОЛОШКИТЕ n -ГРУПИ

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Резиме

Во работава се докажува следнава

Теорема. Нека Q е тополошка n -група, а G групата што е слободна покривка на Q . Ако \mathcal{B} е фамилијата подмножества на G определена со:

$$\mathcal{B} = \{A_1 \cdots A_k \mid A_1, \dots, A_k \text{ се отворени во } Q\}$$

тогаш имаме:

(i) \mathcal{B} е база на топологија \mathcal{G} над G , при што дадената топологија над Q е индуцирана од \mathcal{G} . Q е и отворено и затворено подмножество во G .

(ii) G е тополошка група во однос на топологијата \mathcal{G} .

(iii) Ако тополошката n -група Q е компактна (Хаусдорфова), тогаш и тополошката група G е компактна (Хаусдорфова).

(Притоа, за алгебрата Q со три $n+1$ -арни операции $(\dots)[\dots]$, $[\dots]$ се вели дека е n -група, ако се исполнети идентитетите (2) и (3); групата G е слободна покривка на n -групата Q ако се исполнети условите (4), (5) и (6); n -групата Q се вика тополошка ако Q е тополошки простор, таков што операциите на n -групата се непрекинати.)