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ZEROLESS INTERVALS OF REAL POLYNOMIALS

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ABSTRACT. *Interval* $\beth \subset \mathbb{R} \cup \{\pm\infty\}$ *is obtained by elementary considerations on which the normalized real polynomial*

$$\mathcal{P}_n(x) = x^n + p_{n-1}x^n + \cdots + p_1 x - p_0,\ p_0 \geq 0$$

does not have any zero.

The well known Laplace and Cauchy theorems precize the upper/lower bounds of zeros for real polynomial $\mathcal{P}_n(x)$ in terms of its coefficients, polynomial degree n (and maximal index of negative coefficients - Cauchy's theorem). That means $m, M > 0$ are given for which $|\mathcal{P}_n(x)| > 0$ if $x \notin (m, M)$, see e.g. [1, Chapter 29]. Actually, by these theorems one deduces an annulus $[m', m]$, $mm' \leq 0$ on which $|\mathcal{P}_n(x)| > 0$. In this short note we will be interested in zeroless interval $\beth = [a, b]$, $0 \leq a \leq 1 \leq b \leq \infty$ in which a real polynomial cannot vanish, compare Theorem 1; moreover the Remark 1 contains links to zeroless interval for the negative real halfline. These intervals we obtain elementary.

Consider a real polynomial $\mathcal{P}_n(x) = x^n + p_{n-1}x^n + \cdots + p_1 x - p_0$, $p_0 \geq 0$. Let $\mathbb{J}^- := \{j : p_j < 0\}$, $\mathbb{J}^+ := \{j : p_j > 0\}$ and assume the following structure of index sets:

$$\mathbb{J}^- := \{\mathfrak{i}, \cdots, \mathfrak{j}\}, \quad \mathfrak{i} < \cdots < \mathfrak{j}; \qquad \mathbb{J}^+ := \{\nu, \cdots, n\}, \quad \nu < \cdots < n.$$

Then by

$$p_j = \begin{cases} p_j^+ & j \in \mathbb{J}^+ \\ p_j^- & j \in \mathbb{J}^- \\ 0 & j \in \{0, 1, \cdots, n\} \setminus (\mathbb{J}^- \cup \mathbb{J}^+) \end{cases}$$

we get the representation

$$\mathcal{P}_n(x) = \sum_{j \in \mathbb{J}^+} p_j^+ x^j - \sum_{j \in \mathbb{J}^-} p_j^- x^j - p_0. \tag{1}$$

Let $\mathfrak{c} > 0$ be a positive zero of $\mathcal{P}_n(x)$, i.e. $\mathcal{P}_n(\mathfrak{c}) = 0$. It is easy to see that

$$\sum_{j \in \mathbb{J}^-} p_j^- \mathfrak{c}^j \leq \max\{\mathfrak{c}^{\mathfrak{i}}, \mathfrak{c}^{\mathfrak{j}}\} \sum_{j \in \mathbb{J}^-} p_j^-, \qquad \sum_{j \in \mathbb{J}^+} p_j^+ \mathfrak{c}^j \geq \min\{\mathfrak{c}^{\nu}, \mathfrak{c}^n\} \sum_{j \in \mathbb{J}^+} p_j^+. \tag{2}$$

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Applying (2) to the identity $\mathcal{P}_n(\mathfrak{c}) = 0$ we clearly deduce

$$\min\{\mathfrak{c}^\nu, \mathfrak{c}^n\} \sum_{j\in\mathbb{J}^+} p_j^+ \le p_0 + \max\{\mathfrak{c}^{\mathfrak{i}}, \mathfrak{c}^{\mathfrak{j}}\} \sum_{j\in\mathbb{J}^-} p_j^- \,. \tag{3}$$

Two cases arise.

(i) $\mathfrak{c} \in (0,1)$. With this assumption (3) becomes

$$\mathfrak{c}^n \sum_{j\in\mathbb{J}^+} p_j^+ \le p_0 + \mathfrak{c}^{\mathfrak{i}} \sum_{j\in\mathbb{J}^-} p_j^- < p_0 + \sum_{j\in\mathbb{J}^-} p_j^- ,$$

i.e.

$$\mathfrak{c} < \left(\frac{p_0 + \sum_{j\in\mathbb{J}^-} p_j^-}{\sum_{j\in\mathbb{J}^+} p_j^+} \right)^{1/n} . \tag{4}$$

(ii) $\mathfrak{c} > 1$. Now, by (3) we arrive at

$$\sum_{j\in\mathbb{J}^+} p_j^+ < \mathfrak{c}^\nu \sum_{j\in\mathbb{J}^+} p_j^+ \le p_0 + \mathfrak{c}^{\mathfrak{j}} \sum_{j\in\mathbb{J}^-} p_j^- < \mathfrak{c}^{\mathfrak{j}} \Big(p_0 + \sum_{j\in\mathbb{J}^-} p_j^- \Big).$$

Hence

$$\mathfrak{c} > \left(\frac{\sum_{j\in\mathbb{J}^+} p_j^+}{p_0 + \sum_{j\in\mathbb{J}^-} p_j^-} \right)^{1/\mathfrak{j}} := \boldsymbol{L} \,. \tag{5}$$

Exchanging the evaluation directions in (2) we get

$$p_0 + \min\{\mathfrak{c}^{\mathfrak{i}}, \mathfrak{c}^{\mathfrak{j}}\} \sum_{j\in\mathbb{J}^-} p_j^- \le \max\{\mathfrak{c}^\nu, \mathfrak{c}^n\} \sum_{j\in\mathbb{J}^+} p_j^+ . \tag{6}$$

(iii) $\mathfrak{c} \in (0,1)$. By (6) we deduce

$$\mathfrak{c}^{\mathfrak{j}} \Big(p_0 + \sum_{j\in\mathbb{J}^-} p_j^- \Big) < p_0 + \mathfrak{c}^{\mathfrak{j}} \sum_{j\in\mathbb{J}^-} p_j^- \le \mathfrak{c}^\nu \sum_{j\in\mathbb{J}^+} p_j^+ < \sum_{j\in\mathbb{J}^+} p_j^+ ,$$

which results in upper bound $\mathfrak{c} < \boldsymbol{L}$ and finally

(iv) $\mathfrak{c} > 1$. This case gives us

$$\mathfrak{c}^n \sum_{j\in\mathbb{J}^+} p_j^+ \ge p_0 + \mathfrak{c}^{\mathfrak{i}} \sum_{j\in\mathbb{J}^-} p_j^- > p_0 + \sum_{j\in\mathbb{J}^-} p_j^- ,$$

which means $\mathfrak{c} > \boldsymbol{L}^{-\mathfrak{j}/n}$.

Collecting all these estimates upon $\mathfrak{c}$ we deduce the following result.

Theorem 1. *Let $\mathfrak{c} > 0$ be a zero of $\mathcal{P}_n(x) = x^n + p_{n-1}x^n + \cdots + p_1 x - p_0$, $p_0 \ge 0$. Then no other zeros of this polynomial are contained in the interval*

$$\beth = \Big[\min\{\boldsymbol{L}, \boldsymbol{L}^{-\mathfrak{j}/n}\}, \max\{\boldsymbol{L}, \boldsymbol{L}^{-\mathfrak{j}/n}\} \Big] , \tag{7}$$

where $\mathfrak{j}$ *is the maximal element in index set of negative coefficients of* $\mathcal{P}_n(x)$ *and*

$$\boldsymbol{L} := \left(\frac{\sum_{j\in\mathbb{J}^+} p_j^+}{p_0 + \sum_{j\in\mathbb{J}^-} p_j^-} \right)^{1/\mathfrak{j}} \tag{8}$$

Remark 1. *The case of negative zeros we can handle considering* $\mathcal{P}_n(-\mathfrak{c}) = 0,\ \mathfrak{c} > 0$ *in above exposed manner. At first we get* (1), *after that we arrive at* $\boldsymbol{L}$ *and endly in interval* $\beth$ *which is precised by Theorem* 1.

Remark 2. *It is clear that the sign of coefficient* p_0 *is not really of majorant importance. We assume that the initial polynomial is of the form*

$$\mathcal{P}_n(x) = x^n + p_{n-1}x^n + \cdots + p_1 x - p_0, \qquad p_0 \geq 0.$$

But $p_0 \leq 0$ *is not a restriction at all, because then the coefficient* p_0 *will be added to* $\sum_{j\in\mathbb{J}^+} p_j^+$ *in* (8).

Finally, for specific cases of real, positive coefficients ($\beth = \emptyset$), symmetric polynomials - Schur theorem, etc. consult [1, Chapters 7, 29, 31].

Example 1. Consider the polynomial

$$\mathcal{P}_5(x) = x^5 + 2x^3 - 7x^2 - 2x - 3.$$

As $n = 5$, $\mathfrak{j} = 2$, $\sum_{j\in\mathbb{J}^+} p_j^+ = 1+2 = 3$, $\sum_{j\in\mathbb{J}^-} p_j^- = 7+2 = 9$, we get by (8) that

$$\boldsymbol{L} = \sqrt{\frac{3}{12}} = \frac{1}{2}, \qquad \boldsymbol{L}^{-2/5} = \sqrt[5]{4}\,.$$

By the Theorem 1, the polynomial has no zeros in $[0.5, \sqrt[5]{4}]$. Indeed, the *Mathematica 5.0* gives one real zero $x_0 \approx 1.77017$.

References

[1] G. Kurepa. *Higher Algebra* vol. **I**, **II**. Zavod za izdavanje udžbenika, Beograd, 1971. (in Serbo-Croatian).

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