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ON THE LIE GROUPS SO(3) AND SO(4)

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In this paper a description of the local Lie group for SO(3) is given, and all the matrices of SO(3) are also obtained. This implies the well known fact that SO(3) is homeomorphic to RP^3. A description of the local Lie group for SO(4) is also giv.

1. Introduction

Let us consider the following 2x2 matrices

$$A_{11} = A_{22} = \frac{1-t^2}{1+t^2}$$
$$A_{12} = -A_{21} = \frac{2t}{1+t^2} \tag{1.1}$$

where $t \in R$, and the negative unit matrix $-I_{2x2}$ corresponding to $t=\pm\infty$. Then we obtain a set of matrices which are continuous with respect to each of the four elements, where $t \in RP^1$. This set with the matrix multiplication is in fact the Lie group SO(2). Now the following question appears naturally: Is it possible to give a similar description of the local Lie group for SO(n), and is it possible to obtain all the matrices of SO(n)? We shall give an answer to these questions in the next two sections for n=3 and n=4.

2. The structure of the Lie group SO(3)

Let $P_x, P_y, P_z \in R$. Then it is easy to verify that the following 3x3 matrix

$$A_{11} = (1+P_x^2-P_y^2-P_z^2)/\alpha$$
$$A_{22} = (1-P_x^2+P_y^2-P_z^2)/\alpha$$
$$A_{33} = (1-P_x^2-P_y^2+P_z^2)/\alpha$$
$$A_{12} = 2(P_z+P_xP_y)/\alpha \tag{2.1}$$
$$A_{23} = 2(P_x+P_yP_z)/\alpha$$
$$A_{31} = 2(P_y+P_zP_x)/\alpha$$

$$A_{21} = 2(-P_z+P_xP_y)/\alpha$$
$$A_{32} = 2(-P_x+P_yP_z)/\alpha$$
$$A_{13} = 2(-P_y+P_zP_x)/\alpha$$

where $\alpha=1+P_x^2+P_y^2+P_z^2$, is an element of SO(3). If $P_x=P_y=P_z=0$ we obtain the unit matrix, and A_{ij} are continuous functions of P_x, P_y, P_z. Moreover, SO(3) is 3-dimensional Lie group and so the matrices in (2.1) contain a neighbourhood of the unit element in the group SO(3).

Suppose that $P_x, P_y, P_z \to \pm\infty$ such that $P_x:P_y:P_z=a_x:a_y:a_z$. Then (2.1) yield to the following matrix of SO(3)

$$\begin{aligned} B_{11} &= (a_x^2-a_y^2-a_z^2)/\alpha \\ B_{22} &= (-a_x^2+a_y^2-a_z^2)/\alpha \\ B_{33} &= (-a_x^2-a_y^2+a_z^2)/\alpha \\ B_{12} &= B_{21} = 2a_xa_y/\alpha \\ B_{13} &= B_{31} = 2a_xa_z/\alpha \\ B_{23} &= B_{32} = 2a_ya_z/\alpha \end{aligned} \tag{2.2}$$

where $\alpha=a_x^2+a_y^2+a_z^2\neq 0$.

Now we will prove that if (P_x,P_y,P_z) and (P_x',P_y',P_z') determine the same matrix of the type (1.1), then $P_x'=P_x$, $P_y'=P_y$ and $P_z'=P_z$. Indeed, the vectors (P_x',P_y',P_z') and (P_x,P_y,P_z) are collinear to the vector $(A_{23}-A_{32},A_{31}-A_{13},A_{12}-A_{21})$ and the moduls P and P' of the vectors (P_x,P_y,P_z) and (P_x',P_y',P_z') are uniquely determined by the following equalities

$$A_{11} + A_{22} + A_{33} = (3-P^2)/(1+P^2)$$

and

$$A_{11} + A_{22} + A_{33} = (3-P'^2)/(1+P'^2).$$

Now it follows that $(P_x',P_y',P_z') = (P_x,P_y,P_z)$.

It is easy to verify that if (a_x,a_y,a_z) and (a_x',a_y',a_z') determine the same matrix of the type (2.2), then there exists $k\neq 0$ such that $(a_x',a_y',a_z')=k(a_x,a_y,a_z)$. Moreover, the matrices of the types (2.1) and (2.2) belong to disjoint sets because these of the first type are not symmetric except the unit matrix, but these of the second type are symmetric. So there exists a bijection between RP^3 and the set M of the matrices of the first and the second type. The topology on RP^3 induces a topology on M, and all nine matrix components are continuous with respect to that topology. Since M is a compact set, it follows that (M,*), where * is the matrix multiplication, is a Lie group, which is obviously isomorphic to SO(3).

3. Description of the local Lie group of SO(4)

Let us suppose that $P_x,P_y,P_z,Q_x,Q_y,Q_z \in R$. Then one can verify that the following 4x4 matrix

$$\begin{aligned}
A_{11} &= (1+P_x^2-P_y^2-P_z^2-Q_x^2+Q_y^2+Q_z^2-S^2)/\alpha \\
A_{22} &= (1-P_x^2+P_y^2-P_z^2+Q_x^2-Q_y^2+Q_z^2-S^2)/\alpha \\
A_{33} &= (1-P_x^2-P_y^2+P_z^2+Q_x^2+Q_y^2-Q_z^2-S^2)/\alpha \\
A_{44} &= (1+P_x^2+P_y^2+P_z^2-Q_x^2-Q_y^2-Q_z^2-S^2)/\alpha \\
A_{12} &= 2(P_z+SQ_z-Q_xQ_y+P_xP_y)/\alpha \\
A_{23} &= 2(P_x+SQ_x-Q_yQ_z+P_yP_z)/\alpha \\
A_{31} &= 2(P_y+SQ_y-Q_xQ_z+P_xP_z)/\alpha \\
A_{21} &= 2(-P_z-SQ_z-Q_xQ_y+P_xP_y)/\alpha \\
A_{32} &= 2(-P_x-SQ_x-Q_yQ_z+P_yP_z)/\alpha \\
A_{13} &= 2(-P_y-SQ_y-Q_xQ_z+P_xP_z)/\alpha \\
A_{14} &= 2(-Q_x-SP_x+P_yQ_z-P_zQ_y)/\alpha \\
A_{24} &= 2(-Q_y-SP_y+P_zQ_x-P_xQ_z)/\alpha \\
A_{34} &= 2(-Q_z-SP_z+P_xQ_y-P_yQ_x)/\alpha \\
A_{41} &= 2(Q_x+SP_x+P_yQ_z-P_zQ_y)/\alpha \\
A_{42} &= 2(Q_y+SP_y+P_zQ_x-P_xQ_z)/\alpha \\
A_{43} &= 2(Q_z+SP_z+P_xQ_y-P_yQ_x)/\alpha
\end{aligned} \tag{3.1}$$

where $\alpha=1+P_x^2+P_y^2+P_z^2+Q_x^2+Q_y^2+Q_z^2+S^2$ and $S=P_xQ_x+P_yQ_y+P_zQ_z$, is an element of SO(4). If $P_x=P_y=P_z=Q_x=Q_y=Q_z=0$ then we obtain the unit matrix, and A_{ij} are continuous functions of P_x,P_y,P_z,Q_x,Q_y,Q_z. Moreover, SO(4) is 6-dimensional Lie group and so the matrices in (3.1) contain a neighbourhood of the unit element in the group SO(4).

From (3.1) some other classes of orthogonal 4x4 matrices can be obtained. For example, if $P_x, P_y, P_z \in R$ and $Q_x, Q_y, Q_z \to \pm\infty$ such that $Q_x:Q_y:Q_z = a_x:a_y:a_z$, then we obtain the following matrix

$$
\begin{aligned}
B_{11} &= (-a_x^2+a_y^2+a_z^2-S^2)/\alpha \\
B_{22} &= (a_x^2-a_y^2+a_z^2-S^2)/\alpha \\
B_{33} &= (a_x^2+a_y^2-a_z^2-S^2)/\alpha \\
B_{44} &= -1 \\
B_{12} &= 2(Sa_z-a_xa_y)/\alpha \\
B_{23} &= 2(Sa_x-a_ya_z)/\alpha \\
B_{31} &= 2(Sa_y-a_za_x)/\alpha \\
B_{21} &= 2(-Sa_z-a_xa_y)/\alpha \\
B_{32} &= 2(-Sa_x-a_ya_z)/\alpha \\
B_{13} &= 2(-Sa_y-a_za_x)/\alpha \\
B_{14} &= B_{24} = B_{34} = B_{41} = B_{42} = B_{43} = 0
\end{aligned}
\tag{3.2}
$$

where $\alpha=a_x^2+a_y^2+a_z^2+S^2$ and $S=a_xP_x+a_yP_y+a_zP_z$. If $Q_x, Q_y, Q_z \in R$ and $P_x, P_y, P_z \to \pm\infty$ such that $P_x:P_y:P_z=a_x:a_y:a_z$, then we obtain the following matrix

$$
\begin{aligned}
C_{11} &= (a_x^2-a_y^2-a_z^2-S^2)/\alpha \\
C_{22} &= (-a_x^2+a_y^2-a_z^2-S^2)/\alpha \\
C_{33} &= (-a_x^2-a_y^2+a_z^2-S^2)/\alpha \\
C_{44} &= (a_x^2+a_y^2+a_z^2-S^2)/\alpha \\
C_{12} &= C_{21} = 2a_xa_y/\alpha \\
C_{23} &= C_{32} = 2a_ya_z/\alpha \\
C_{31} &= C_{13} = 2a_xa_z/\alpha \\
C_{14} &= -C_{41} = -2a_xS/\alpha \\
C_{24} &= -C_{42} = -2a_yS/\alpha \\
C_{34} &= -C_{43} = -2a_zS/\alpha
\end{aligned}
$$

where $\alpha=a_x^2+a_y^2+a_z^2+S^2$ and $S=a_xQ_x+a_yQ_y+a_zQ_z$.

If S=0, then the limits $\lim_{v\to v_o} A_{ij}$, i,j=1,2,3,4, where $v = (P_x^2+P_y^2+P_z^2, Q_x^2+Q_y^2+Q_z^2, S)$ and $v_o=(\infty,\infty,0)$, do not exist, and so we are not able to determine now the topology of SO(4) as it was done for the group SO(3).

However it is known that the fundamental group for SO(4) is Z_2 and SU(2)⊗SU(2) is its universal covering group, and SU(2) is homeomorphic to the sphere S^3 ([1], [2]).

REFERENCES

[1] Gilmore R., "Lie Groups, Lie Algebras, and Some of Their Applications", A Wiley-interscience publication, New York, 1974

[2] Постников М.М., "Группы и алгебры Ли", Наука, Москва, 1982

ЗА ЛИЕВИТЕ ГРУПИ SO(3) И SO(4)

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Р е з и м е

Во овој труд се дава опис на сите матрици од Лиевата група SO(3) од којшто се гледа дека таа Лиева група е хомеоморфна со RP^3. При тој хомеоморфизам меѓу RP^3 и SO(3), на векторите од $R^3\setminus\{(0,0,0)\}$ одговараат несиметричните матрици од SO(3), на нултиот вектор одговара единичната матрица, а на бескрајните точки од RP^3 одговараат симетричните матрици од SO(3). Исто така даден е опис на локалната Лиева група за SO(4), како и некои класи од ортогонални 4x4 матрици чија детерминанта е 1.