Математички Билтен
13 (XXXIX)
1989 (53-56)
Скопје, Југославија

GEOMETRICAL INTERPRETATION OF THE ANTISYMMETRIC COVARIANT DIFFERENTIATION

Kostadin Trenčevski

Abstract. In this paper it is given a geometrical interpretatiot of the differentiation (1), where A is a tensor field of type (r,s). Indeed it is proved the equality (5) where ϕ_o^D denotes the parallel displacement along the curve $\tau_4^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$, and $\xi^r=\delta^r_k$, $\eta^s=\delta^s_p$ (fig. 1).

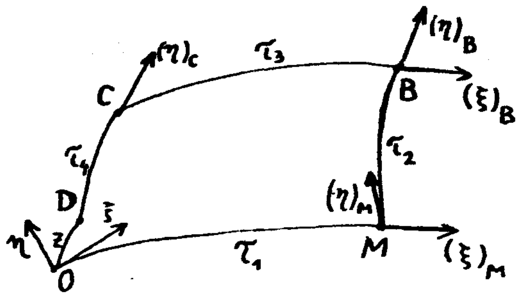

Fig. 1

Let M be an n-dimensional manifold which is endowed with a linear connection. We choose an arbitrary point O∈M and a local coordinate system $x^1,\dots,x^n$ in a neighbourhood of O. We shall denote by C^2 the set of all functions which are twice differentiable and the second derivative is a continuous function in that neighbourhood with respect to the coordinate system (x^i).

Let $A^{i_1\dots i_r}_{j_1\dots j_s}\in C^2$ be the tensor field of type (r,s), then

$$(DA)^{i_1\dots i_r}_{j_1\dots j_s kp} = A^{i_1\dots i_r}_{j_1\dots j_s;k;p} - A^{i_1\dots i_r}_{j_1\dots j_s;p;k} \tag{1}$$

defines a differentiation which is antisymmetric with respect to the indices k and p. In this paper we will give a geometrical interpretation for this differentiation. Using the Ricci identity, (1) can be written in the following form

$$(DA)^{i_1\ldots i_r}_{j_1\ldots j_s kp} = -R^{i_1}{}_{qkp}A^{qi_2\ldots i_r}_{j_1\ldots j_s} - \ldots - R^{i_r}{}_{qkp}A^{i_1\ldots q}_{j_1\ldots j_s} + \\ + R^q{}_{j_1 kp}A^{i_1\ldots i_r}_{qj_2\ldots j_s} + \ldots + R^q{}_{j_s kp}A^{i_1\ldots i_r}_{j_1\ldots q} + T^q_{pk}A^{i_1\ldots i_r}_{j_1\ldots j_s;q}. \tag{2}$$

Let $\xi,\eta\in T_O(M)$ (fig. 1), and let $\tau_1(t)$ $(0\le t\le\Delta u)$, $\tau_2(t)$ $(0\le t\le\Delta v)$, $\tau_3(t)$ $(0\le t\le\Delta u)$ and $\tau_4(t)$ $(0\le t\le\Delta v)$ be geodesic lines which connect O and M, M and B, B and C, C and D respectively, such that

$$(d\tau_1(t)/dt)_{t=o} = \xi, \qquad (d\tau_2(t)/dt)_{t=o} = (\eta)_M,$$
$$(d\tau_3(t)/dt)_{t=o} = -(\xi)_B, \qquad (d\tau_4(t)/dt)_{t=o} = -(\eta)_C.$$

In the above formulas $(\eta)_M$ is the vector at the point M which is obtained by parallel displacement of η along the curve τ_1, $(\xi)_B$ is the vector at the point B which is obtained by parallel displacement of ξ along the curve $\tau_1\cdot\tau_2$, and $(\eta)_C$ is the vector at the point C which is obtained by parallel displacement of η along the curve $\tau_1\cdot\tau_2\cdot\tau_3$.

Now we are going to find the differences of the coordinates between the points D and O to the second order of approximation with respect to Δu and Δv. Since τ_1 is a geodesic line, it holds

$$(x^k)_M - (x^k)_O = (d\tau_1^k(t)/dt)_{t=o}\Delta u + \tfrac{1}{2}(d^2\tau_1^k(t)/dt^2)_{t=o}(\Delta u)^2 + \\ + o(\Delta u^2) = \xi^k\Delta u - \tfrac{1}{2}\Gamma^k_{rs}\xi^r\xi^s(\Delta u)^2 + o(\Delta u^2).$$

Similarly we can calculate the differences $(x^k)_B-(x^k)_M$, $(x^k)_C-(x^k)_B$ and $(x^k)_D-(x^k)_C$, and one can obtain that

$$(x^k)_D - (x^k)_O = T^k_{rs}\xi^r\eta^s\Delta u\Delta v + o(\Delta u^2) + o(\Delta v^2). \tag{3}$$

Specially, if $\xi^r=\delta^r_k$ and $\eta^s=\delta^s_p$, (3) implies that

$$(x^r)_D - (x^r)_O = T^r_{kp}\Delta u\Delta v + o(\Delta u^2) + o(\Delta v^2). \tag{4}$$

<u>Theorem</u>. Let $A\in C^2$ be a tensor field in a neighbourhood of the point O. Then the following formula holds

$$(DA)^{i_1\dots i_r}_{j_1\dots j_s pk} = \lim_{\substack{\Delta u\to o\\ \Delta v\to o}} \frac{1}{\Delta u\Delta v}\left[\phi^D_O(A^{i_1\dots i_r}_{j_1\dots j_s})_D - (A^{i_1\dots i_r}_{j_1\dots j_s})_O\right] \qquad (5)$$

where ϕ^D_O denotes the parallel displacement along the curve $\tau_4^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$, and $\xi^r=\delta^r_k$, $\eta^s=\delta^s_p$ (fig. 1).

<u>Proof</u>. It is well known that if a tensor A at any point of the manifold is transported parallelly round an infinitesimal circuit in the form of an infinitesimal parallelogram with adjacent edges dx^i and δx^i, then the principal part of the infinitesimal increment in $A^{i_1\dots i_r}_{j_1\dots j_s}$ is

$$\begin{aligned}\Delta A^{i_1\dots i_r}_{j_1\dots j_s} &= (-A^{q\dots i_r}_{j_1\dots j_s} R^{i_1}{}_{qkp} - \dots - A^{i_1\dots q}_{j_1\dots j_s} R^{i_r}{}_{qkp} + \\ &+ A^{i_1\dots i_r}_{q\dots j_s} R^q{}_{j_1kp} + \dots + A^{i_1\dots i_r}_{j_1\dots q} R^q{}_{j_skp}) dx^k\delta x^p.\end{aligned} \qquad (6)$$

Let us consider now the closed curve ODCBMO (fig. 1), where the points O and D are connected with a smooth curve z(t). It can be considered as a small distortion of an infinitesimal parallelogram with adjacent edges $\eta^i=\delta^i_p$ and $\xi^i=\delta^i_k$. Let ϕ^O_D denotes the parallel displacement from O to D along the curve z(t). The formula (6) can be applied and we obtain

$$\phi^O_D(\phi^D_O(A^{i_1\dots i_r}_{j_1\dots j_s})_D) - (A^{i_1\dots i_r}_{j_1\dots j_s})_D =$$

$$= -\left[(DA)^{i_1\dots i_r}_{j_1\dots j_s kp} + T^q_{kp} A^{i_1\dots i_r}_{j_1\dots j_s;q}\right]\Delta u\Delta v + o(\Delta u^2) + o(\Delta v^2).$$

Applying the formula for the parallel displacement of the tensor $\phi^D_O(A^{i_1\dots i_r}_{j_1\dots j_s})$ from O to D, and using that

$$(A^{i_1\dots i_r}_{j_1\dots j_s})_D = (A^{i_1\dots i_r}_{j_1\dots j_s})_O + \partial(A^{i_1\dots i_r}_{j_1\dots j_s})/\partial x^k((x^k)_D-(x^k)_O) + o(\Delta u^2) + o(\Delta v^2)$$

we obtain

$$\phi^D_O(A^{i_1\dots i_r}_{j_1\dots j_s})_D - (A^{i_1\dots i_r}_{j_1\dots j_s})_O = (DA)^{i_1\dots i_r}_{j_1\dots j_s pk}\Delta u\Delta v + o(\Delta u^2) + o(\Delta v^2).$$

If we devide this equality by $\Delta u \Delta v$ and put $\Delta u \to 0$, $\Delta v \to 0$ we will obtain the formula (5). ||

The above theorem gives the required geometrical interpretation of the antisymmetric differentiation D, using the parallel displacement of tensors. It is well known that the covariant differentiation can be written in the following form

$$A^{i_1 \dots i_r}_{j_1 \dots j_s ; k} = \lim_{\Delta u \to o} \frac{1}{\Delta u}\left[\phi^O_M (A^{i_1 \dots i_r}_{j_1 \dots j_s}) - (A^{i_1 \dots i_r}_{j_1 \dots j_s})_M\right],$$

where ϕ^O_M denotes the parallel displacement from O to M along the geodesic line $\tau_1(t)$ $(0 \le t \le \Delta u)$ (fig. 1). We notice that both of these differentiations have similar geometrical interpretations.

Similar result can be obtained if one does not use indices. Indeed, if $A \in C^2$ is a tensor field in a neighbourhood of the point O and $X, Y \in T_O(M)$, then the following formula

$$D_{Y,X}(A) = \lim_{\substack{\Delta u \to o \\ \Delta v \to o}} \frac{1}{\Delta u \Delta v}\left[\phi^D_O (A_D) - A_O\right] \tag{7}$$

holds, where

$$D_{Y,X} = \nabla_X \cdot \nabla_Y - \nabla_Y \cdot \nabla_X + \nabla_{(\nabla_Y X)} - \nabla_{(\nabla_X Y)} .$$

We note that $\Delta u / \Delta v$ and $\Delta v / \Delta u$ in (5) and (7) are supposed to be bounded numbers.

R E F E R E N C E S

[1] L.P.Eisenhart: "Riemannian Geometry", Princeton University

[2] S.Kobayashi and K.Nomizu: "Foundations of differential geometry", Volume I, Interscience Publishers, New York, 1963

[3] K.Trenčevski: "On a proof of the Ricci identity", God. zbor. Matem. fak. 33-34, 1982-1983, p. 59-62

ГЕОМЕТРИСКА ИНТЕРПРЕТАЦИЈА НА АНТИСИМЕТРИЧНОТО КОВАРИЈАНТНО ДИФЕРЕНЦИРАЊЕ

Костадин Тренчевски

Р е з и м е

Во овој труд е дадена геометриска интерпретација на диференцирањето дефинирано со (1) односно (2), каде што A е тензорно поле од тип (r,s). Всушност, се докажува равенството (5) каде што ϕ^D_O означува паралелно пренесување по должината на кривата $\tau_4^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$, и $\xi^r = \delta^r_k$, $\eta^s = \delta^s_p$ (сл. 1).