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ON SOME CONSEQUENCES OF THE GENERALIZED GAUSS-BONNET THEOREM

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Abstract. In this paper two consequences of the generalized Gauss-Bonnet theorem are obtained. It is proved in theorem 1 that if M^{2n} is a compact and orientable differentiable manifold such that $\chi(M^{2n})\neq 0$ and $B_{2q}=0$, then there does not exist an orientable subbundle of the tangent bundle with fiber R^{2q} on M^{2n}. In theorem 2, it is proved that if ∂M^{2n+1} is a compact manifold and $\chi(\partial M^{2n+1})\neq 0$, where M^{2n+1} is an orientable differentiable manifold, then M^{2n+1} does not admit an orientable subbundle E of the tangent bundle with fiber R^{2n} such that the restriction of E on ∂M^{2n+1} is the tangent bundle of ∂M^{2n+1}. Two consequences of these two theorems are obtained.

Characteristic classes are subject of consideration of topology, but they can be considered from a view point of differential geometry, using the differential forms [2]. Using the Gauss-Bonnet theorem and dealing with differential forms, in this paper two theorems are obtained, which were not known to the author before. As an immediate corollary of the theorem 1, we obtain in this paper a proof of the problem 9C (p.102-103) in [3]. Theorem 3 is a consequence of the theorem 2.

It is well known that if M^{2n} is a compact and orientable differentiable manifold such that $\chi(M^{2n})\neq 0$, then there does not exist an orientable subbundle of the tangent bundle with fiber R^{2q+1}. In theorem 1 we shall see what happens with the orientable subbundles of the tangent bundle with fiber R^{2q}.

We will denote by B_r (r-th Betti number) the rank od the group $H_r(M^{2n},Z)$, and by χ the Euler characteristic.

Theorem 1. Let M^{2n} be a compact and orientable differentiable manifold such that $\chi(M^{2n})\neq 0$. If $B_{2q}=0$, then there does not exist an orientable subbundle of the tangent bundle with fiber R^{2q} on M^{2n}.

Proof. Suppose that the assumptions of the theorem are satisfied and assume that E is an orientable subbundle of the tangent bundle with fiber R^{2q}. Since M^{2n} is an orientable manifold, there exists a complementary orientable subbundle E′ of the tangent bundle with fiber R^{2n-2q}. Let w and w′ be the corresponding closed differential forms which represent the Euler characteristic classes for E and E′. Since $B_{2q}=0$, it follows that w is an exact differential form, i.e. $w=d\phi$, and hence

$$\int_{M^{2n}} w\wedge w' = \int_{M^{2n}} (d\phi)\wedge w' = \int_{M^{2n}} d(\phi\wedge w') = \int_{\partial M^{2n}} \phi\wedge w' = 0.$$

On the other hand, $w\wedge w'$ is a closed differential form which represents the Euler characteristic class for the tangent bundle on M^{2n}, and the generalized Gauss-Bonnet theorem implies that

$$\int_{M^{2n}} w\wedge w' = \chi(M^{2n}) \neq 0.$$

This contradiction proves the theorem. ||

Corollary. If $q\neq 0$ and $q\neq n$, then the sphere S^{2n} does not admit an orientable subbundle of the tangent bundle with fiber R^{2q}.

Theorem 2. Suppose that M^{2n+1} is an orientable differentiable manifold such that ∂M^{2n+1} is a compact manifold and $\chi(\partial M^{2n+1})\neq 0$. Then M^{2n+1} does not admit an orientable subbundle E of the tangent bundle with fiber R^{2n}, such that the restriction of E on ∂M^{2n+1} is the tangent bundle of ∂M^{2n+1}.

Proof. Suppose that the assumptions of the theorem are satisfied, and assume that there exists an orientable subbundle E of the tangent bundle of M^{2n+1} with fiber R^{2n}, such that the restriction of E on ∂M^{2n+1} is the tangent bundle of ∂M^{2n+1}. Let w be the corresponding closed differential 2n-form which represents the Euler characteristic class for E. Then we obtain a contradiction, because

$$0 = \int_{M^{2n+1}} dw = \int_{\partial M^{2n+1}} w = \chi(\partial M^{2n+1}) \neq 0,$$

and this proves the theorem. ||

Let D^2 be a 2-dimensional disc, and let a continuous vector field A^i of non-zero tangent vectors of $S^1=\partial D^2$ be given. It is well known([5], p.288)that the vector field A^i can not be continuated to a continuous vector field on D^2 which is non-zero at each point. Using the theorem 2, now we shall obtain a similar result.

Theorem 3. Let M^{2n+1} be a differentiable manifold such that ∂M^{2n+1} is a compact manifold and $\chi(\partial M^{2n+1})\neq 0$. If ∂M^{2n+1} admits 2n continuous vector fields $A^i_{(1)},\ldots,A^i_{(2n)}$, which are linearly independent at each point, then these 2n vector fields can not continuously be continuated over M^{2n+1} such that they are linearly independent at each point of M^{2n+1}.

Proof. Suppose that $B^i_{(1)},\ldots,B^i_{(2n)}$ are continuous vector fields which are linearly independent at each point of M^{2n+1} and such that their restrictions on ∂M^{2n+1} are the vector fields $A^i_{(1)},\ldots,A^i_{(2n)}$. It follows ([6]) that M^{2n+1} is an orientable manifold. The vector fields $B^i_{(1)},\ldots,B^i_{(2n)}$ generate an orientable subbundle E of the tangent bundle with fiber R^{2n} on M^{2n+1}, and the restriction of E on M^{2n+1} is the tangent bundle on ∂M^{2n+1}. This contradicts the theorem 2. ||

R E F E R E N C E S

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ЗА НЕКОИ ПОСЛЕДИЦИ ОД ГЕНЕРАЛИЗИРАНАТА ГАУС-БОНЕТ-ОВА ТЕОРЕМА

Костадин Тренчевски

Р е з и м е

Во овој труд се добиени две последици од генерализираната Гаус-Бонет-ова теорема. Во теорема 1 се докажува дека: ако M^{2n} е компактно и ориентабилно диференцијабилно многуобразие такво што $\chi(M^{2n})\neq 0$ и $B_{2q}=0$, тогаш не постои ориентабилно подраслојување од тангентното раслојување со слој R^{2q} на M^{2n}. Во теорема 2 се докажува дека: ако ∂M^{2n+1} е компактно многуобразие и $\chi(\partial M^{2n+1})\neq 0$, каде што M^{2n+1} е ориентабилно диференцијабилно многуобразие, тогаш M^{2n+1} не допушта ориентабилно подраслојување E од тангентното раслојување со слој R^{2n}, така што рестрикцијата од E врз ∂M^{2n+1} да биде тангентното раслојување на ∂M^{2n+1}. Во трудот се добиени уште две последици од овие две теореми.