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FREE GRUPOIDS WITH $x^2x^2 = x^3x^3$

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Abstract. A description of free objects in the variety $\mathcal{V}$ of groupoids defined by the identity $x^2x^2 = x^3x^3$ is obtained. The following method is used: one of the sides of the identity is considered as "suitable" and the other as "unsuitable" one. First, the left-hand side x^2x^2 is chosen as "suitable" and the set of elements of F (**F** being an absolutely free groupoid with a basis B) containing no parts that have the form x^3x^3 is taken as a "candidate" for the carrier of the desired free object in $\mathcal{V}$. Continuing this procedure, a $\mathcal{V}$-free object is obtained. Another construction of $\mathcal{V}$-free object is obtained by choosing the right-hand side x^3x^3 as "suitable" one.

0. Introduction

First, we introduce some notations.

Throughout the paper, $\boldsymbol{F} = (F, \cdot)$ will denote a given absolutely free groupoid[2)] (i.e. groupoid free in the class of all groupoids) with the basis B. The following two properties characterize $\boldsymbol{F}$ ([1]; L.1.5):

a) $\boldsymbol{F}$ *is injective* (i.e. $ab = cd \Rightarrow a = c, b = d$);

b) *The set B of primes*[3)] *is nonempty and generates* $\boldsymbol{F}$.

For every $w \in F$, a set $P(w)$ (called the *set of parts* of w) and the *length* $|w|$ of w are defined by:

$$P(b) = \{b\}, \quad P(uv) = \{uv\} \cup P(u) \cup P(v), \quad |b| = 1, \quad |uv| = |u| + |v|,$$

for every $b \in B$ and $u, v \in F$.

The subject of this paper is a construction of free groupoids in the variety $\mathcal{V}$ of groupoids defined by the identity

$$x^2x^2 = x^3x^3. \text{ }^{4)} \tag{0.1}$$

In order to construct $\mathcal{V}$-free objects (i.e. free objects in the variety $\mathcal{V}$) we will recall the corresponding procedure given in [2] for the variety $\mathcal{V}_1$ of groupoids defined by the identity

$$xx^2 = x^2x^2 \tag{0.2}$$

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[2]Notions as: groupoid, free groupoid, homomorphism ... have the usual meanings.

[3]In a groupoid $\mathbf{G} = (G, \cdot)$, $a \in G$ is *prime* iff $a \neq xy$, for all $x, y \in G$.

[4]Here, x^n is defined by: $x^1 = x$, $x^{k+1} = x^k x$.

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Namely, choose first x^2x^2 as the "suitable" (i.e. xx^2 as the "unsuitable") side of (0.2). As a "candidate" for the carrier of a $\mathcal{V}_1$-free groupoid, define the set

$$R = \{t \in F : (\forall \alpha \in F)\ \alpha\alpha^2 \notin P(t)\},$$

and then define an operation $*$ on R by

$$t, u \in R \Rightarrow \{t * u = tu \text{ if } tu \in R \ \&\ t * u = (t^2)^2 \text{ if } u = t^2\}.$$

The obtained groupoid $\boldsymbol{R} = (R, *)$ is a $\mathcal{V}_1$-free groupoid with the basis B.

Next, we choose xx^2 as the "suitable" (i.e. x^2x^2 as the "unsuitable") side and define a "candidate" for the carrier of a $\mathcal{V}_1$-free groupoid to be the set

$$F_1 = \{t \in F : (\forall \alpha \in F)\ (\alpha^2)^2 \notin P(t)\},$$

and then an operation $*_1$ on F_1 by

$$t, u \in F_1 \Rightarrow \{t *_1 u = tu \text{ if } tu \in F_1 \ \&\ t *_1 u = \alpha\alpha^2 \text{ if } t = u = \alpha^2\}.$$

Then $\boldsymbol{F}_1 = (F_1, *_1)$ is a groupoid that is not in $\mathcal{V}_1$. As a consequence of the identity $xx^2 = x^2x^2$, we come to a new identity $\alpha^2(\alpha\alpha^2) = (\alpha\alpha^2)^2$. This suggests a definition of a new "candidate" $\boldsymbol{F}_2 = (F_2, *_2)$:

$$F_2 = \{t \in F_1 : (\forall \alpha \in F_1)(\alpha\alpha^2)^2 \notin P(t)\},$$

We obtain that $\boldsymbol{F}_2 \notin \mathcal{V}_1$ and come to a new identity in $\mathcal{V}_1$:

$$(\alpha\alpha^2)(\alpha^2(\alpha\alpha^2)) = ((\alpha^2(\alpha\alpha^2))^2.$$

Continuing this procedure, we see regularity in the consequences of (0.2) that suggests introducing a special kind of groupoid powers $x \mapsto x^{<n>}$ defined by:

$$x^{<0>} = x, \quad x^{<1>} = x^2, \quad x^{<k+2>} = x^{<k>}x^{<k+1>}. \tag{0.3}$$

Using this, we have: $(\alpha^2)^2 = (\alpha^{<1>})^2$, $(\alpha\alpha^2)^2 = (\alpha^{<2>})^2$ etc. and a sequence of groupoids $\boldsymbol{F}_n = (F_n, *_n)$, $n \geq 0$, defined by: $\boldsymbol{F}_0 = \boldsymbol{F} = (F, \cdot)$,

$$F_1 = \{t \in F_0 : (\forall \alpha \in F_0)(\alpha^{<1>})^2 \notin P(t)\},$$
$$F_n = \{t \in F_{n-1} : (\forall \alpha \in F_n)(\alpha^{<n>})^2 \notin P(t)\},$$

$$t, u \in F_n \Rightarrow \{t *_n u = t *_{n-1} u \text{ if } t *_{n-1} u \in F_n \ \&\ t *_n u = \alpha^{<n+1>} \text{ if } t = u = \alpha^{<n>}\}$$

The groupoids $\boldsymbol{F}_n$ are not in $\mathcal{V}_1$.However, the fact that $F \supseteq F_1 \cdots \supseteq F_n \supseteq \dots$ and that $\boldsymbol{F}_n$ is "better" than $\boldsymbol{F}_{n-1}$ enables us to define a carrier R' of a free object in $\mathcal{V}_1$ by:

$$R' = \{t \in F : (\forall \alpha \in F,\ k \geq 1)\ (\alpha^{<k>})^2 \notin P(t)\}\ \left(= \bigcap\{F_n : n \geq 1\}\right)$$

and an operation $*'$ on R' by:

$$t, u \in R' \Rightarrow \{t *' u = tu \text{ if } tu \in R' \ \&\ t *' u = \alpha^{<k+1>} \text{ if } t = u = \alpha^{<k>},\ k \geq 1\}.$$

Then $\boldsymbol{R}' = (R', *')$ is a $\mathcal{V}_1$-free groupoid with the basis B and it is isomorphic to $\boldsymbol{R}$.

We use below the same method for constructing free objects in the variety $\mathcal{V}$ of groupoids with $x^2x^2 = x^3x^3$.

1. Construction of $\mathcal{V}$-free objects by choosing x^2x^2 as the "suitable side"

Choosing the left-hand side of (0.1) as "suitable", we define the first "candidate" for the carrier of a $\mathcal{V}$-free groupoid by:

$$F_1 = \{t \in F : (\forall \alpha \in F)\ (\alpha^3)^2 \notin P(t)\} \tag{1.1}$$

By (1.1) we obtain:

1) $t, u \in F_1 \Rightarrow \{tu \notin F_1 \Leftrightarrow t = u \text{ is a cube }^5\}$
2) $t, u \in F_1 \Rightarrow \{tu \in F_1 \Leftrightarrow [t \neq u \text{ or } (t = u \text{ is not a cube}]\}$
3) $t^2 \in F_1 \Leftrightarrow \{t \in F_1\ \&\ t \text{ is not a cube}\}$
4) $t^3 \in F_1 \Leftrightarrow t^2 \in F_1$

Define an operation $*_1$ on F_1 by:

$$t, u \in F_1 \Rightarrow t *_1 u = \begin{cases} tu, & \text{if } tu \in F_1 \\ (\alpha^2)^2, & \text{if } t = u = \alpha^3. \end{cases}$$

By a direct verification we obtain that $\boldsymbol{F}_1 = (F_1, *_1)$ is a groupoid. However, the equality (0.1), which has the form here

$$(t *_1 t) *_1 (t *_1 t) = ((t *_1 t) *_1 t) *_1 ((t *_1 t) *_1 t) \tag{1.2}$$

is not satisfied in $\boldsymbol{F}_1$. Namely, for $t = \alpha^3$, the left-hand side of (1.2) is $((\alpha^2)^2)^2$ and the right-hand side is $((\alpha^2)^2\alpha^3)^2$. Thus, $\boldsymbol{F}_1 \notin \mathcal{V}$. Therefore, as a consequence of (1.2), we obtain that: $((\alpha^2)^2)^2 = ((\alpha^2)^2\alpha^3)^2$ is an identity in $\mathcal{V}$.

This suggests a definition of a new "candidate" $\boldsymbol{F}_2 = (F_2, *_2)$:

$$F_2 = \{t \in F_1 : (\forall \alpha \in F_1)\ ((\alpha^2)^2\alpha^3)^2 \notin P(t)\},$$

$$t, u \in F_2 \Rightarrow t *_2 u = \begin{cases} t *_1 u, & \text{if } t *_1 u \in F_2 \\ ((\alpha^2)^2)^2, & \text{if } t = u = (\alpha^2)^2\alpha^3. \end{cases}$$

Checking (1.2) (when $*_1$ is substituted by $*_2$), we obtain that $\boldsymbol{F}_2 \in \mathcal{V}$ and one more identity in $\mathcal{V}$: $(((\alpha^2)^2)^2)^2 = (((\alpha^2)^2)^2((\alpha^2)^2\alpha^3))^2$. Continuing this procedure, we can see "regularity" in the consequences of the identity (1.2). This suggests introducing the following notations:

$$\begin{aligned} x^{(0)} &= x, \quad x^{(k+1)} = \left(x^{(k)}\right)^2; \\ x^{[0]} &= x, \quad x^{[k+1]} = x^{(k+1)}x^{[k]}. \end{aligned} \tag{1.3}$$

It is easily seen, by induction on n, that:

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[5] $a \in G$ is a *cube* in a grupoid $\mathbf{G} = (G, \cdot)$ iff $(\exists \alpha \in G) a = \alpha^3$; if $\mathbf{G}$ is injective then α is unique.

Proposition 1.1. *If* $\boldsymbol{G} = (G, \cdot)$ *is any groupoid, then for each* $x \in G$ *and* $m, n \geq 0$:

$$x^{(m+n)} = \left(x^{(m)}\right)^{(n)}.$$

By induction on p, one can show the following propositions:

Proposition 1.2. *If* $x, y \in F$ *and* $p, q \geq 0$, *then:*

$a)$ $|x^{(p)}| = 2^p|x|$; $b)$ $|x^{[p]}| = (2^{p+1} - 1)|x|$

$c)$ $x^{(p)} = y^{(p)} \Leftrightarrow x = y$; $d)$ $x^{(p)} = y^{(p+q)} \Leftrightarrow x = y^{(q)}$;

$e)$ $(\forall x \in F)\ (\exists!(y, p) \in F \times \mathbb{N}_0)[x = y^{(p)}\ \&\ (\forall z \in F) y \notin z^2]$ [6)]

$f)$ $x^{[p+1]} = y^{[q+1]} \Rightarrow p = q,\ x = y$.

Proposition 1.3. *If* $\boldsymbol{G} = (G, \cdot) \in \mathcal{V}$, *then for each* $x \in G$ *and* $p, q \geq 0$:

$$\left(x^{[p]}\right)^2 = \left(x^{(p)}\right)^2.$$

More generally: $\left(x^{[p]}\right)^{(r)} = x^{(p+r)}$, *for any* $r \geq 1$.

Proof. . Clearly, the above equality holds for $p = 0$. Suppose that it is true for $p = k$. Then, considering the identity (0.1) and the inductive hypothesis, we have:

$$(x^{[k+1]})^2 = (x^{(k+1)}x^{[k]})^2 = ((x^{(k)})^2 x^{[k]})^2 = ((x^{[k]})^2 x^{[k]})^2 = ((x^{[k]})^3)^2 =$$
$$((x^{[k]})^2)^2 = ((x^{(k)})^2)^2 = (x^{(k+1)})^2.$$

□

Using (1.3), we can define the following infinite set of groupoids:

$$F_1 = \{t \in F : (\forall \alpha \in F)\ (\alpha^{[1]})^2 \notin P(t)\},$$

$$t, u \in F_1 \Rightarrow t *_1 u = \begin{cases} tu, & \text{if } tu \in F_1 \\ \alpha^{(2)}, & \text{if } t = u = \alpha^{[1]}. \end{cases}$$

$$F_{n+1} = \{t \in F_n : (\forall \alpha \in F_n)\ (\alpha^{[n+1]})^2 \notin P(t)\},$$

$$t, u \in F_{n+1} \Rightarrow t *_{n+1} u = \begin{cases} t *_n u, & \text{if } t *_n u \in F_{n+1} \\ \alpha^{(n+2)}, & \text{if } t = u = \alpha^{[n+1]}. \end{cases}$$

One can show that $\boldsymbol{F}_{n+1}$ is a groupoid and $\boldsymbol{F}_{n+1} \notin \mathcal{V}$.

The fact that $F \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \ldots$ and that $\boldsymbol{F}_{n+1}$ is "better" than $\boldsymbol{F}_n$ suggests to define the carrier of a free groupoid in $\mathcal{V}$ in the following way:

$$R = \{t \in F : (\forall \alpha \in F, k \geq 1)\ (\alpha^{[k]})^2 \notin P(t)\}. \tag{1.4}$$

(Note that it is not necessary to define the whole sequence, since the desired "good candidate" can be noticed after several steps.)

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[6]$\mathbb{N}_0$ is the set of nonnegative integers.

By (1.4) we obtain:

0) $B \subset R \subset F$

i) $t, u \in R \Rightarrow \{tu \notin R \Leftrightarrow (\exists\alpha \in F, k \geq 1)\, t = u = \alpha^{[k]}\}$

ii) $t, u \in R \Rightarrow \{tu \in R \Leftrightarrow [t \neq u \text{ or } (t = u \ \& \ (\forall\alpha \in R, k \geq 1)\, t \neq \alpha^{[k]})]\}$

iii) $t^{(p+1)} \in R \Leftrightarrow t \in R \ \& \ t \neq \alpha^{[k]},\ k \geq 1.$

Theorem 1. *Let R be defined by* (1.4) *and an operation* $*$ *on R by:*

$$t, u \in R \Rightarrow \{t * u = tu \ \text{ if } \ tu \in R \ \& \ t * u = \alpha^{(k+1)} \ \text{ if } \ t = u = \alpha^{[k]}\}.$$

Then $\boldsymbol{R}= (R, *)$ *is a* $\mathcal{V}$*-free groupoid with the basis B.*

Proof. It follows that, for every $u \in F$, there is at most one pair $(\alpha, k) \in F \times \mathbb{N}_0$, such that $u = \alpha^{[k]}$. By a direct verification of (0.1) we obtain that $R \in \mathcal{V}$. Furthermore, B is a generating set of R and for any groupoid $G \in \mathcal{V}$ and a mapping $\lambda : B \to G$ there is a homomorphism $\varphi : \boldsymbol{R} \to \boldsymbol{G}$ that extends λ. □

2. Construction of $\mathcal{V}$-free objects if x^3x^3 is the "suitable side"

Now, choose the right-hand side of (0.1) as "suitable" and define:

$$F_1' = \{t \in F : (\forall\alpha \in F)\, (\alpha^2)^2 \notin P(t)\}. \tag{2.1}$$

By (2.1) we obtain:

1′) $t, u \in F_1' \Rightarrow \{tu \notin F_1' \Leftrightarrow t = u$ is a square $\}$[7]

2′) $t, u \in F_1' \Rightarrow \{tu \in F_1' \Leftrightarrow [t \neq u$ or $(t = u$ is not a square $]\}$

3′) $t^2 \in F_1' \Leftrightarrow \{t \in F_1' \ \& \ t$ is not a square$\}$

4′) $t^2 \in F_1' \Leftrightarrow t^n \in F_1',\ n \geq 3.$

Define an operation $*_1'$ on F_1' by:

$$t, u \in F_1' \Rightarrow t *_1' u = \begin{cases} tu, & \text{if } \ tu \in F_1' \\ (\alpha^3)^2, & \text{if } \ t = u = \alpha^2. \end{cases}$$

By a direct verification we obtain that $\boldsymbol{F}_1{}' = (F_1', *_1')$ is a groupoid. However, the equality

$$(t *_1' t) *_1' (t *_1' t) = ((t *_1' t) *_1' t) *_1' ((t *_1' t) *_1' t) \tag{2.2}$$

is not satisfied in $\boldsymbol{F}_1{}'$. Namely, for $t = \alpha^2$, the left-hand side of (2.2) is $((\alpha^3)^2\alpha^2)^2$ and the right-hand side is $((\alpha^3)^3)^2$. Thus, $\boldsymbol{F}_1{}' \notin \mathcal{V}$. Therefore, as a consequence of (0.1), we obtain that: $((\alpha^3)^2\alpha^2)^2 = ((\alpha^3)^3)^2$ is an identity in $\mathcal{V}$.

This suggests a definition of a new "candidate" $\boldsymbol{F}_2{}' = (F_2', *_2')$:

$$F_2' = \{t \in F_1' : (\forall\alpha \in F_1')\, ((\alpha^3)^2\alpha^2)^2 \notin P(t)\}$$

[7] $a \in G$ is a square in a groupoid $\mathbf{G}= (G, \cdot)$ iff $(\exists\alpha \in G)\ a = \alpha^2$; if G is injective, then α is unique.

$$t, u \in F_2' \Rightarrow t *_2' u = \begin{cases} t *_1' u, & \text{if } tu \in F_1' \\ \left((\alpha^3)^3\right)^2, & \text{if } t = u = (\alpha^3)^2\alpha^2. \end{cases}$$

Checking (2.2) (when $*_1'$ is substituted by $*_2'$), we obtain that $\boldsymbol{F_2}' \notin \mathcal{V}$ and one more identity in $\mathcal{V}$: $(((\alpha^3)^3)^2((\alpha^3)^2\alpha^2))^2 = (((\alpha^3)^3)^3)^2$.

Continuing this procedure, we can see a "regularity" in the consequences of the identity (2.2). This suggests introducing the following notations:

$$x^{<0>} = x, \qquad x^{<k+1>} = (x^{<k>})^3;$$
$$x^{<0]} = x^2, \qquad x^{<k+1]} = (x^{<k+1>})^2 x^{<k]} \tag{2.3}$$

It is easily seen, by induction on n, that:

Proposition 2.1. *If $\boldsymbol{G} = (G, \cdot)$ is any groupoid, then for each $x \in G$ and $m, n \geq 0$:*

$$x^{<m+n>} = (x^{<m>})^{<n>}.$$

By induction on p, one can show the following propositions:

Proposition 2.2. *If $x, y \in F$ and $p, q \geq 0$, then:*

a) $|x^{<p>}| = 3^p|x|$

b) $x^{<p>} = y^{<p+q>} \Leftrightarrow x = y^{<q>}$;

c) $(\forall x \in F)(\exists!(y,p) \in F \times \mathbb{N}_0)[x = y^{<p>}$ & *y is not a cube*].

Proposition 2.3. *If $x, y \in F$ and $p, q \geq 0$, then:*

a) $|x^{<p]}| = (3^{p+1} - 1)|x|$; *b)* $|x^{<p]}| < |x^{<p+1>}|$; *c)* $x^{<p]} \neq x^{<p+m>}, m \geq 1$;

d) $x^{<p+1>} \neq y^{<q]}$, $p \geq 0, q \geq 1$; *e)* $x^{<p]} = y^{<q]} \Rightarrow p = q,\ x = y$.

Proposition 2.4. *If $\boldsymbol{G} = (G, \cdot) \in \mathcal{V}$, then for each $x \in G$ and $p, q \geq 0$:*

$$\left(x^{<p]}\right)^2 = \left(x^{<p+1>}\right)^2.$$

More generally: $(x^{<p]})^{<r>} = x^{<p+r>}$, *for any $r \geq 1$.*

As in (1.4), we define the carrier of a free groupoid in $\mathcal{V}$ in the following way:

$$R' = \{t \in F : \ (\forall \alpha \in F, k \geq 0)\ (\alpha^{<k]})^2 \notin P(t)\}. \tag{2.4}$$

By (2.4) we obtain:

$0'$) $B \subset R' \subset F$

i') $t, u \in R' \Rightarrow \{tu \notin R \Leftrightarrow (\exists \alpha \in F)\ t = u = \alpha^{<k]}, k \geq 0\}$

ii') $t, u \in R' \Rightarrow \{tu \in R' \Leftrightarrow \left[t \neq u \text{ or } (t = u \,\&\, (\forall \alpha \in F, k \geq 0)\, t \neq \alpha^{<k]})\right]\}$

Theorem 2. *Let R' be defined by (2.4) and an operation $*'$ on R' by:*

$$t, u \in R' \Rightarrow \{t *' u = tu \ \ \text{if} \ \ tu \in R' \ \& \ t *' u = (\alpha^{<k+1>})^2 \ \ \text{if} \ \ t = u = \alpha^{<k]}\}.$$

Then $\boldsymbol{R'} = (R', *')$ is a $\mathcal{V}$-free groupoid with the basis B.

Proof. It follows that, for every $u \in F$ there is at most one pair $(\alpha, k) \in F \times \mathbb{N}_0$, such that $u = \alpha^{<k]}$. By a direct verification of (0.1) we obtain that $\boldsymbol{R}' \in \mathcal{V}$. Furthermore, B generates $\boldsymbol{R}'$ and, for any $\boldsymbol{G} \in \mathcal{V}$ and a mapping $\lambda : B \to G$, there is a homomorphism $\varphi : \boldsymbol{R} \to \boldsymbol{G}$ that extends λ. □

(Note that $\boldsymbol{R}$ and $\boldsymbol{R}'$ are isomorphic with the same basis B.)

3. Some remarks

Remark 3.1: The method used above is not applicable in some varieties of groupoids. Namely, consider the variety of groupoids with the identity $x^2 = x^3$. If we choose x^2 as the "suitable side" of the identity and define

$$R = \{t \in F : (\forall \alpha \in F)\ \alpha^3 \notin P(t)\},$$

$$t, u \in R \Rightarrow \{t * u = tu \ \text{ if } \ tu \in R \,\&\, t * u = u^2 \ \text{ if } \ t = u^2\},$$

then we obtain that $\boldsymbol{R} = (R, *)$ is a free object in this variety. However, if we choose the right-hand side as the "suitable" one, then by

$$R' = \{t \in F : (\forall \alpha \in F)\ \alpha^2 \notin P(t)\},$$

$$t, u \in R' \Rightarrow \{t *' u = tu \ \text{ if } \ tu \in R' \,\&\, t *' u = t^3 \ \text{ if } \ t = u\},$$

$R' = (R', *')$ is not a groupoid. (Namely, if $t = u$, then $t *' t = t^3 = t^2 t \notin P(t)$!)

Thus, the procedure used for the variety $\mathcal{V}$ is not applicable in one of the cases for the variety of groupoids with the identity $x^2 = x^3$ and in any variety of groupoids with the identity such that one hand-side of the identity is a part of the other one.

Remark 3.2: It is natural to consider the "shorter" side of the identity $x^2x^2 = x^3x^3$ as a "suitable" one (as we did in Section 1) and to expect a "shorter" (or a "less complicated") construction of a free groupoid in this variety. However, comparing the constructions in Section 1 and Section 2 we can see that they are nearly equal, although one can say that the groupoid powers (1.3) are a "little simpler" than (2.3). Moreover, the situation with the variety defined by $xx^2 = x^2x^2$ is quite different. Namely, the choice of the "shorter" side xx^2 as "suitable" leads to a longer and more complicated construction than the choice of the "larger" side x^2x^2 (the construction in this case finishes at once, at the first step!). (Probably, the "symmetry" in x^2x^2 plays a certain role.)

Remark 3.3: The free groupoids $\boldsymbol{R}$ and $\boldsymbol{R}'$ obtained in Theorems 1 and 2 are $\mathcal{V}$-canonical groupoids. (A groupoid $\boldsymbol{H} = (H, *)$ is said to be $\mathcal{V}$-canonical groupoid in a given variety $\mathcal{V}$([3]) iff:

(c_0) $B \subset H \subset F$ $\quad (c_1)$ $tu \in H \Rightarrow t, u \in H \,\&\, tu = t * u$; $\quad (c_2)$ $\boldsymbol{H}$ is $\mathcal{V}$-free

(i.e. $\boldsymbol{H} \in \mathcal{V}$; B generates $\boldsymbol{H}$; for any $\boldsymbol{G} \in \mathcal{V}$ and any mapping $\lambda : B \to G$, there is a homomorphism φ from $\boldsymbol{F}$ into $\boldsymbol{G}$ such that $\varphi_B = \lambda$).

For a given variety $\mathcal{V}$ of groupoids, a set R is said to be *representative* for $\mathcal{V}$ ([4]) iff the following conditions are satisfied:

(j_0) $R \subseteq F$;

(j_1) for every $w \in F$ there is exactly one $u \in F$ such that $u \in R$ and the equation (w, u) is satisfied in $\mathcal{V}$;

(j_2) if $t \in R$, then $P(t) \subseteq R$.

Proposition 3.1. *The carrier of any $\mathcal{V}$-canonical groupoid is a representative set for $\mathcal{V}$.*

Proof. Let $\mathcal{V}$ be a variety of groupoids and $\boldsymbol{R} = (R, *)$ be a $\mathcal{V}$- canonical groupoid (with a basis B). If $\boldsymbol{F}$ is an absolutely free groupoid with a basis B, then there is a unique homomorphism φ from $\boldsymbol{F}$ into $\boldsymbol{R}$ such that $\varphi(b) = b$, for any $b \in B$. Therefore, for every $w \in F$, $\varphi(w)$ is a uniquely determined element of R and clearly the equation $(w, \varphi(w))$ is satisfied in $\mathcal{V}$. Thus, (j_1) holds. The condition (j_2) can be shown by induction on length of t. Namely, if $|t| = 1$, i.e. $t \in B$, then $P(t) = \{t\} \subseteq R$. Suppose that $P(t) \subseteq R$ for every $t \in R$ with $|t| \leq k$. Let $t \in R$ be such that $|t| = k+1$. Then $t = uv$, $|u| \leq k, |v| \leq k$ and since $\{uv\}, P(u), P(v) \subseteq R$, it follows that $P(t) \subseteq R$. □

References

[1] R.H.Bruck: *A Survey of Binary Systems*, Springer-Verlag 1958

[2] Ǵ. Čupona, V.Celakoska-Jordanova: *On a variety of groupoids of rank* 1; Proceed. 2nd Congress of Math. and Inf. of R.Macedonia , Ohrid 2000, 17-23

[3] Ǵ. Čupona, N. Celakoski, B. Janeva: *Injective Groupoids in some Varieties of Groupoids*, Proceed. 2nd Congress of Math. and Inf. of R.Macedonia , Ohrid 2000, 47-55

[4] Ježek: *Free Groupoids In Varieties Determined by a Short Equation*, Acta Universitatis Carolinae - Math. et Phys., Vol.23. No.1 (1982), 3-24

СЛОБОДНИ ГРУПОИДИ СО $x^2x^2 = x^3x^3$

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Резиме

Во работава е даден опис на слободните објекти во многуобразието $\mathcal{V}$ од групоиди дефинирано со идентитетот $x^2x^2 = x^3x^3$. Користена е следнава постапка: едната од двете страни на идентитетот ја сметаме за "соодветна", а другата за "несоодветна". Разгледани се двата можни случаи. Прво, левата страна x^2x^2 е земена за "соодветна". Во тој случај, множеството елементи од F (каде што $\boldsymbol{F}$ е апсолутно слободен групоид со база B) коишто не содржат дел од обликот x^3x^3, земено е како "кандидат" за носител на слободен објект во $\mathcal{V}$. Продолжувајќи ја таа постапка, добиен е $\mathcal{V}$-слободен групоид. Друг $\mathcal{V}$-слободен групоид е конструиран со земање на десната страна x^3x^3 како "соодветна". (Добиените $\mathcal{V}$-слободни групоиди се изоморфни.) Меѓутоа, оваа постапка не е применлива во некои многуобразија групоиди, како на пример во многуобразието дефинирано со идентитетот $x^2 = x^3$, а и во секое многуобразие групоиди со идентитет во кој едната страна е дел од другата

(Remark 3.1). Добиените $\mathcal{V}$-слободни групоиди $\boldsymbol{R}$ и $\boldsymbol{R'}$ (Theorem 1 и Theorem 2) се $\mathcal{V}$-канонични. Се покажува (Proposition 3.1) дека *носителот* на *$\mathcal{V}$-каноничен групоид е репрезентативно множество за $\mathcal{V}$* (Remark 3.3)

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