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(3,2)-SEMIGROUPS GENERATED BY (3,2)-AUTOMATA

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Abstract. In this paper we give a generalization of the classical result that any finite automata generates a finite semigroup. We give an algorithm for generating finite $(3,2)$-semigroups by a finite $(3,2)$-automata.

1. Preliminaries

The notion of vector valued semigroup automaton is introduced in [1]. Here we will concentrate on vector valued $(3,2)-$semigroup automata, and the semigroups generated by them.

Let B be a nonempty set, called **alphabet**, and his elements called **letters**.

A nonempty set S together with a map $f : S \times B^2 \to S \times B$ is called $(3,2)$-**automaton** and is denoted by (S, B, f). The set S is called **set of states**, and f is called **transition function** of the automata (S, B, f).

The clasical notion of semigroup automata is the following. Let $(B, \cdot)$ be a semigroup. A nonempty set S together with a map $f : S \times B \to S$ such that

$$f(f(s, x), y) = f(s, x \cdot y), \quad \text{for every } s \in S \text{ and } x, y \in B$$

is called **semigroup automaton** and is denoted by $(S, (B, \cdot), f)$.

A map $\{\} : B^3 \to B^2$ is called **(3,2)-operation**, and the pair $(B, \{\})$ is called **(3,2)-groupoid**.

A (3,2)-groupoid $(B, \{\})$ is called **(3,2)-semigroup** if $\{x\{yzt\}\} = \{\{xyz\}t\}$, for all $x, y, z, t \in B$.

A nonempty set S together with a map $f : S \times B^2 \to S \times B$ is called **(3,2)-automaton** and is denoted by (S, B, f). It is called **finite** if both the sets B and S are finite.

Let $(B, \{\})$ be a (3,2)-semigroup. A nonempty set S together with a map $f : S \times B^2 \to S \times B$ such that

$$f(f(s, x, y)z) = f(s, \{xyz\}), \text{ for every } s \in S \text{ and } x, y, z \in B,$$

is called **(3,2)-semigroup automaton** and is denoted by $(S, (B, \{\}), f)$.

The aim of this paper is to show how a given finite (3,2)-automaton (S, B, f) generates a (3,2)-semigroup, and its associated (3,2)-semigroup automaton.

From now on, all the (3,2)-automata will be finite.

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2. (3,2)-GROUPOIDS GENERATED BY (3,2)-AUTOMATA

In the definition of a (3,2)-automaton f is only a map from $S \times B^2$ to $S \times B$. In order f to be a transition function of a (3,2)-semigroup automaton, it is necessary to be:

$$f(f(s,x,y)z) = f(s,\{xyz\}), \text{ for every } s \in S \text{ and } x,y,z \in B, \tag{1}$$

where $(B,\{\})$ has to be a (3,2)-semigroup, i.e. the (3,2)-associativity

$$\{\{xyz\}t\} = \{x\{yzt\}\}, \text{ for all } x,y,z,t \in B \tag{2}$$

has to be satisfied. All this, means that for any $x,y,z \in B$, there has to exist a pair $(u,v) \in B^2$ such that

$$f(f(s,x,y)z) = f(s,u,v), \text{ for every } s \in S. \tag{$*$}$$

Let (S,B,f) is an (3,2)-automaton given by a table. We extend this table with new columns, denoted by $\{xyz\}$, and filling them up according to the condition (1). The condition $(*)$ says that the columns of every triple $\{xyz\}$ has to coincide with the column of some pair (u,v). But, it is possible to have more than one pair, or not to have a pair satisfying the above condition. So our discussion goes in this direction.

If for each triplet $\{xyz\}$ for $x,y,z \in B$ there is at least one pair satisfying (1), then we can define a (3,2)-operation by: $\{xyz\} = (u,v)$, i.e. at least one (3,2)-groupoid $(B,\{\})$ is generated.

Example 1: Let $S = \{s_0,s_1,s_2\}$, $B = \{a,b\}$ and the map f be given by the Table 1.

f	(a,a)	(a,b)	(b,a)	(b,b)
s_0	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)
s_1	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)
s_2	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)

Table 1

We calculate:

$f(f(s_0,a,a),a) = f(s_2,b,a) = (s_1,b)$
$f(f(s_1,a,a),a) = f(s_1,b,a) = (s_2,a)$
$f(f(s_2,a,a),a) = f(s_2,b,a) = (s_1,b)$,

and from Table 1 it can be seen that the pairs $(s_1,b),(s_2,a),(s_1,b)$ are obtained via the transition of the state $s_i, i = 0,1,2$ by the pair (a,b) or (b,a) , i.e.

$f(s_0,a,b) = f(s_0,b,a) = (s_1,b)$;
$f(s_1,a,b) = f(s_1,b,a) = (s_2,a)$;
$f(s_2,a,b) = f(s_2,b,a) = (s_1,b)$.

So, $f(f(s_0,a,a),a) = f(s_0,a,b) = f(s_0,b,a)$
$f(f(s_1,a,a),a) = f(s_1,a,b) = f(s_1,b,a)$
$f(f(s_2,a,a),a) = f(s_2,a,b) = f(s_2,b,a)$,

and using the definition of a (3,2)-automata, we have $\{aaa\} = (a,b) = (b,a)$.

We do similar calculation on the other triples, and for the given (3,2)-automata we obtain four (3,2)-groupoids, given in the Tables 1.1 - 1.4.

{ }	
a a a	(b,a)
a a b	(b,b)
a b a	(b,b)
a b b	(a,a)
b a a	(b,b)
b a b	(a,a)
b b a	(a,a)
b b b	(b,a)

Table 1.1

{ }	
a a a	(b,a)
a a b	(b,b)
a b a	(b,b)
a b b	(a,a)
b a a	(b,b)
b a b	(a,a)
b b a	(a,a)
b b b	(a,b)

Table 1.2

{ }	
a a a	(a,b)
a a b	(b,b)
a b a	(b,b)
a b b	(a,a)
b a a	(b,b)
b a b	(a,a)
b b a	(a,a)
b b b	(b,a)

Table 1.3

{ }	
a a a	(a,b)
a a b	(b,b)
a b a	(b,b)
a b b	(a,a)
b a a	(b,b)
b a b	(a,a)
b b a	(a,a)
b b b	(a,b)

Table 1.4

Example 2: Let $S = \{s_0, s_1, s_2\}, B = \{a, b\}$ and let the (3,2)-automaton be given by the Table 2.

f	(a,a)	(a,b)	(b,a)	(b,b)
s_0	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
s_1	(s_0, a)	(s_2, b)	(s_2, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)

Table 2

We calculate:
$f(f(s_0, a, a), a) = f(s_0, a, a) = f(s_0, a)$
$f(f(s_1, a, a), a) = f(s_0, a, a) = f(s_0, a)$
$f(f(s_2, a, a), a) = f(s_2, a, a) = f(s_2, a)$

and from the Table 2 it can be seen that the pairs $(s_0, a), (s_0, a), (s_2, a)$ are obtained via the transition of the state $s_i, i = 0, 1, 2$ by the pair (a, a), and using the definition of a (3,2)-automaton, we have $\{aaa\} = (a, a)$. We do similar calculation on the other triples, and for the given (3,2)-automaton we obtain only one (3,2)-groupoid, given in the Table 2.1.

{ }	
a a a	(a,a)
a a b	(a,a)
a b a	(b,a)
a b b	(a,b)
b a a	(b,a)
b a b	(b,a)
b b a	(b,a)
b b b	(b,b)

Table 2.1

The previous two examples were examples with the existence of more pairs and existence of only one pair.

But it is possible for a given triplet $\{xyz\}$ for $x, y, z \in B$ no pair to satisfy the condition (1), and in that case the given (3,2)-automata does not generate a (3,2)-groupoid. This is the case in the next Example.

Example 3: Let $S = \{s_0, s_1, s_2\}$, $B = \{a, b\}$ and let the (3,2)-automaton be given by the Table 3.

f	(a,a)	(a,b)	(b,a)	(b,b)
s_0	(s_0, a)	(s_0, b)	(s_0, a)	(s_0, b)
s_1	(s_0, a)	(s_2, b)	(s_2, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)

Table 3

We calculate:
$f(f(s_0,a,a),a) = f(s_0,a,a) = (s_0,a)$
$f(f(s_1,a,a),a) = f(s_0,a,a) = (s_0,a)$
$f(f(s_2,a,a),a) = f(s_2,a,a) = (s_2,a)$
and from the Table 3, it can be seen that the pairs $(s_0,a),(s_0,a),(s_2,a)$ are obtained via the transition of the state $s_i, i = 0,1,2$ by the pair (a,a), and using the definition of a (3,2)-automaton, we have $\{aaa\} = (a,a)$.

Next, we calculate,
$f(f(s_0,a,a),b) = f(s_0,a,b) = (s_0,b)$
$f(f(s_1,a,a),b) = f(s_0,a,b) = (s_0,b)$
$f(f(s_2,a,a),b) = f(s_2,a,b) = (s_2,b)$,
and from the Table 3, we see that there is no pair $(u,v), u,v \in B$, satisfying

$$f(f(s_i,a,a),b) = f(s_i,u,v), (\forall s_i \in S),$$

We do similar calculation on the other triplets, and for the given (3,2)-automaton we obtain that the (3,2)-operation for the triples $\{abb\},\{baa\},\{bba\}$ and $\{bbb\}$ are defined, while the (3,2)-operation for the triples $\{aab\},\{aba\},\{bab\}$ can not be defined. So, in this example we obtain the partial (3,2)-operation, given in Table 3.1.

{ }	
a a a	(a,a)
a a b	?
a b a	?
a b b	(a,b)
b a a	(b,a)
b a b	?
b b a	(b,a)
b b b	(b,b)

Table 3.1

3. (3,2)-SEMIGROUPS GENERATED BY (3,2)-AUTOMATA OVER THE ALPHABET B

The discussion in part 2 gives the answer only to the question: when a (3,2)-automata generates a (3,2)-groupoid over the alphabet B. The next aim is to see when the generated (3,2)-groupoid is a (3,2)-semigroup. For this, we have to check the (3,2)-associativity, i.e. the condition (2).

Next, we extend the table of the (3,2)-automaton with new rows (s_i,x), for every $s_i \in S$, $x \in B$ filled up using the map f, defined by:

$$f((s_i,a),(u,v)) = f(f(s_i,a,u),v) \text{ and}$$

$$f((s_i,a),\{xyz\}) = f(f(f(s_i,a,x),y),z). \tag{3}$$

If

$$f((s_i,a),(u,v)) = f((s_i,a),\{xyz\}) \tag{**}$$

for every $a \in B, s_i \in S$, and $\{xyz\} = (u,v)$, i.e. the extended column for the triple $\{xyz\}$ coincides with the extended column for the pair (u,v) then the (3,2)-automaton generates a (3,2)-semigroup. The extended table shows how the (3,2)-semigroup is defined.

In part 2, we have seen that a given (3,2)-automaton can generate more than one (3,2)-groupoids, and if the condition (**) is satisfied for more than one of them, then the (3,2)-automaton generates more than one (3,2)-semigroup.

Example 1': We extend the Table 1 of the (3,2)-automaton from Example 1, with new columns and new rows, and obtain the Table 1' .

	(a,a)	(a,b)	(a,b) (b,a)	(b,b)
s_0	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)
s_1	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)
s_2	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)
(s_0,a)	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)
(s_0,b)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)
(s_1,a)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)
(s_1,b)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)
(s_2,a)	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)
(s_2,b)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)

	(a,b) $\{aaa\}$	(b,b) $\{aab\}$	(b,b) $\{aba\}$	(a,a) $\{abb\}$	(b,b) $\{baa\}$	(a,a) $\{bab\}$	(a,a) $\{bba\}$	(a,b) $\{bbb\}$
s_0	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)
s_1	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)
s_2	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)
(s_0,a)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)
(s_0,b)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)
(s_1,a)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)
(s_1,b)	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)
(s_2,a)	(s_2,a)	(s_2,b)	(s_2,b)	(s_1,b)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)
(s_2,b)	(s_2,b)	(s_1,b)	(s_1,b)	(s_2,a)	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,b)

Table 1'

From Table 1' we see that the column for each triple $\{xyz\}$ coincides with some column of the pairs (u, v), for $u, v \in B$. This means that a (3,2)-semigroup is generated by the (3,2)-automaton over the set B, given by the Table 1.1. Hence the (3,2)-automaton given by the Table 1, is also a (3,2)-semigroup automaton.

4. Extending the set B

If the condition $(*)$ or the condition $(**)$ is not satisfied, then the given (3,2)-automaton does not generate (3,2)-semigroup over the set B. So, the next step is to extend the alphabet B with new letters, one letter for each triple $\{xyz\}$, for which there is no pair (u, v), for $u, v \in B$ satisfying the condition (1), and for each triple $\{xyz\} = (u, v)$ for which the condition $(**)$ is not satisfied. If p_j is a new letter associated to the triple $\{xyz\}$, then to this triple we associate the pair (p_j, p_j), i.e. we define $\{xyz\} = (p_j, p_j)$.

With the above extension we obtain a new alphabet, denoted by B'. The numbers of elements $|B'|$ of the set B' is at most $|B| + |B|^3$, hence it is finite.

Since to some of the triples we associate the new pairs, we extend the table by introducing new columns, denoted by $\{xyzt\}$ for $x, y, z, t \in B$, filled up again by using the condition (3). Then we check if some of the new columns coincide with some of the previous ones, i.e. we check if the condition $(**)$ is satisfied.

If every new column coincides with some of the previous ones, the new alphabet B' is completed, and a (3,2)-semigroup over B' is defined.

Example 2': We extend the Table 2 of the (3,2)-automaton from Example 2 by introducing new columns and rows, and obtain Table 2' and Table 2".

	(a,a)	(a,b)	(b,a)	(b,b)
s_0	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
s_1	(s_0, a)	(s_2, b)	(s_2, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)
(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
(s_0, b)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
(s_1, a)	(s_0, a)	(s_0, a)	(s_2, a)	(s_2, b)
(s_1, b)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)
(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)
(s_2, b)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)

	(a,a) $\{aaa\}$	(a,a) $\{aab\}$	(c,c) $\{aba\}$	(a,b) $\{abb\}$	(b,a) $\{baa\}$	(b,a) $\{bab\}$	(b,a) $\{bba\}$	(b,b) $\{bbb\}$
s_0	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
s_1	(s_0, a)	(s_0, a)	(s_2, a)	(s_2, b)	(s_2, a)	(s_1, a)	(s_1, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)
(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
(s_0, b)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
(s_1, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)
(s_1, b)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)
(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)
(s_2, b)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)

Tabela 2'

From Table 2' we see that for the triple $\{aba\}$ there is no a pair (u, v), for $u, v \in B$, such that $\{aba\} = (u, v)$. So we introduce a new letter c, and extend the alphabet B to the new alphabet. In the next step, we extend Table 2' to Table 2", and we see that all the quadruples $\{xyzt\}$ for $x, y, z, t \in B$ are defined, and so the alphabet B' is defined, and the generated (3,2)-semigroup by the (3,2)-automaton given by Table 2 will be defined over B'.

	(a,a) aaaa	(a,a) aaab	(a,a) aaba	(a,a) aabb	(c,c) abaa	(c,c) abab	(c,c) abba	(a,b) abbb
s_0	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)
s_1	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
s_2	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)
(s_0,b)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)
(s_1,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)
(s_1,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)

	(b,a) baaa	(b,a) baab	(b,a) baba	(b,a) babb	(b,a) bbaa	(b,a) bbab	(b,a) bbba	(b,b) bbbb
s_0	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)
s_1	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
s_2	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)
(s_0,b)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)	(s_0,a)
(s_1,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
(s_1,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)

Tabela 2"

If in the previous extension of B to B', some of the new columns does not coincide by some of the previous ones, then to such a column, we associate new pair (q_j, q_j), where q_j is a new letter. This gives a new alphabet $B"$. Again $|B"|$ is at most $|B'| + |B'|^3$.

We continue this procedure inductevelly, till all the new columns coincide with some of the previous ones.

Example 3': We extend the Table 3 of the (3,2)-automaton from Example 3, with new columns and new rows, and obtain the Table 3'.

	(a,a)	(a,b)	(b,a)	(b,b)
s_0	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
s_1	(s_0,a)	(s_2,b)	(s_2,a)	(s_2,a)
s_2	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
(s_0,a)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_1,a)	(s_0,a)	(s_0,b)	(s_2,a)	(s_2,b)
(s_1,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)

	(a,a) $\{aaa\}$	(c,c) $\{aab\}$	(d,d) $\{aba\}$	(a,b) $\{abb\}$	(b,a) $\{baa\}$	(e,e) $\{bab\}$	(b,a) $\{bba\}$	(b,b) $\{bbb\}$
s_0	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
s_1	(s_0,a)	(s_0,b)	(s_2,a)	(s_2,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
s_2	(s_1,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
(s_0,a)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_1,a)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
(s_1,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)

Tabela 3'

From Table 3' we see that for the triples $\{aab\}$, $\{aba\}$, $\{bab\}$ there are no pairs (u,v), for $u,v \in B$. We associate to them the pairs (c,c), (d,d), (e,e) respectively, where c,d,e are new letters, and the new alphabet is $B' = B \cup \{c,d,e\}$. Next we introduce new columns, and obtain the Table 3".

	(a,a) aaaa	(c,c) aaab	(a,a) aaba	(c,c) aabb	(d,d) abaa	(f,f) abab	(d,d) abba	(a,b) abbb
s_0	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
s_1	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
s_2	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_0,a)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_1,a)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_1,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)

	(b,a) baaa	(e,e) baab	(b,a) baba	(e,e) babb	(b,a) bbaa	(e,e) bbab	(b,a) bbba	(b,b) bbbb
s_0	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
s_1	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
s_2	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
(s_0,a)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)	(s_0,a)	(s_0,b)
(s_1,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)
(s_1,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)
(s_2,b)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,a)	(s_2,b)

Tabela 3"

From Table 3" we see that for the quadruple $\{abab\}$ there is no a pair (u,v), for $u,v \in B'$, so to it we associate the pair (f,f), where f is a new letter, and the new alphabet is $B' = B \cup \{c,d,e,f\}$.

Next, we consider all 5-tuples $\{xyztw\}$, for $x, y, z, t, w \in B$, and obtain that for each of them there is a pair (u, v), for $u, v \in B'$, such that $\{xyztw\} = (u, v)$. Hence the alphabet B' is the one over which the (3,2)-semigroup will be defined.

In the previous discussion we were extending the table by rows, one row for each pair $(s_i, x) \in S \times B$. Since the sets S and B are finite, each column of such extended table has finite length $|S|+|S|\cdot|B|$, and since the elements in the column are from the set $S \times B$, it follows that there are only finitely many different columns (at most $(|S|\cdot|B|)^{|S|+|S|\cdot|B|}$). This implies that after finitely many introductions of new letters, the new columns will coincide with some of the previous ones, and so this procedure will end after finitely many steps. Hence the final alphabet B' will be again finite set.

5. Extending the partial (3,2)-operation

With the previous discussion, we have extended the alphabet to a new alphabet B', over which we are going to define the (3,2)-semigroup. Up to this moment we have only the (3,2)-operation for the triples $\{xyz\}$ for $x, y, z \in B$, and for the triples $\{xtt\}$ and $\{ttx\}$ for $x \in B$ and $t \in B' \setminus B$. This means that from $|B'|^3$ triples we have the (3,2)-operation for $|B|^3 + 2\cdot|B|\cdot(|B'|-|B|)$ triples. Next, we define the (3,2)-operation on the rest $|B'|^3 - [|B|^3 + 2\cdot|B|\cdot(|B'|-|B|)]$ triples, such that the conditions (1) and (2) are satisfied. The partial (3,2)-operation for the (3,2)-automaton from Example 2 is given by Table 2.1', and from Example 3, by table 3.1'

{}	
a a a	(a,a)
a a b	(a,a)
a a c	
a b a	(c,c)
a b b	(a,b)
a b c	
a c a	
a c b	
a c c	(a,a)
b a a	(b,a)
b a b	(b,a)
b a c	
b b a	(b,a)
b b b	(b,b)
b b c	
b c a	
b c b	
b c c	(b,a)
c a a	
c a b	
c a c	
c b a	
c b b	
c b c	
c c a	(c,c)
c c b	(c,c)
c c c	

Table 2.1'

{ }		{ }		{ }		{ }		{ }		{ }	
aaa	(a,a)	baa	(b,a)	caa		daa		eaa		faa	
aab	(c,c)	bab	(e,e)	cab		dab		eab		fab	
aac		bac		cac		dac		eac		fac	
aad		bad		cad		dad		ead		fad	
aae		bae		cae		dae		eae		fae	
aaf		baf		caf		daf		eaf		faf	
aba	(d,d)	bba	(b,a)	cba		dba		eba		fba	
abb	(a,b)	bbb	(b,b)	cbb		dbb		ebb		fbb	
abc		bbc		cbc		dbc		ebc		fbc	
abd		bbd		cbd		dbd		ebd		fbd	
abe		bbe		cbe		dbe		ebe		fbe	
abf		bbf		cbf		dbf		ebf		fbf	
aca		bca		cca	(a,a)	dca		eca		fca	
acb		bcb		ccb	(c,c)	dcb		ecb		fcb	
acc	(c,c)	bcc	(e,e)	ccc		dcc		ecc		fcc	
acd		bcd		ccd		dcd		ecd		fcd	
ace		bce		cce		dce		ece		fce	
acf		bcf		ccf		dcf		ecf		fcf	
ada		bda		cda		dda	(d,d)	eda		fda	
adb		bdb		cdb		ddb	(f,f)	edb		fdb	
adc		bdc		cdc		ddc		edc		fdc	
add	(a,a)	bdd	(b,a)	cdd		ddd		edd		fdd	
ade		bde		cde		dde		ede		fde	
adf		bdf		cdf		ddf		edf		fdf	
aea		bea		cea		dea		eea	(b,a)	fea	
aeb		beb		ceb		deb		eeb	(e,e)	feb	
aec		bec		cec		dec		eec		fec	
aed		bed		ced		ded		eed		fed	
aee	(f,f)	bee	(e,e)	cee		dee		eee		fee	
aef		bef		cef		def		eef		fef	
afa		bfa		cfa		dfa		efa		ffa	(d,d)
afb		bfb		cfb		dfb		efb		ffb	(f,f)
afc		bfc		cfc		dfc		efc		ffc	
afd		bfd		cfd		dfd		efd		ffd	
afe		bfe		cfe		dfe		efe		ffe	
aff	(c,c)	bff	(e,e)	cff		dff		eff		fff	

Table 3.1'

In order to define the required (3,2)-semigroup $(B', \{\})$ in a simpler way, we start with the following definition: $\{xxx\} = (x, x)$ for $x \in B' \setminus B$, if this does not contradict some of the previous definitions, and then we fill up their consequences, using the following conditions:

(i_1) $\{\{xxx\}y\} = \{x\{xxy\}\}$
$\{y\{xxx\}\} = \{\{yxx\}y\}$ for $y \in B'$;

(i_2) $f(s_i, \{xxx\}) = f(f(s_i, x, x), x)$ for every $s_i \in S$.

Next we define $\{abx\}, \{axb\}, \{xab\}$ for $a, b \in B$ and $x \in B' \setminus B$ using the conditions:

(ii_1) $\{y\{abx\}\} = \{\{yab\}x\}$
$\{y\{axb\}\} = \{\{yax\}b\}$
$\{y\{xab\}\} = \{\{yxa\}b\}$, for $y \in B'$;

(ii_2) $f(s_i, \{abx\}) = f(f(s_i, a, b)x)$
$f(s_i, \{axb\}) = f(f(s_i, a, x)b)$
$f(s_i, \{xab\}) = f(f(s_i, x, a)b)$, for every $s_i \in S$;

(ii_3) if there is more than one (u, v) satisfying the conditions (ii_1) and (ii_2), then choose the one for which already exists a previously defined triple $\{xyz\}$.
After the above definitions we fill up their consequences for the (3,2)-operation and the map f.
The (3,2)-semigroups for the (3,2)-automata from Examples 2 and 3 are given by Tables 2.2" and 3.2"

{ }	
aaa	(a,a)
aab	(a,a)
aac	(a,a)
aba	(c,c)
abb	(a,b)
abc	(c,c)
aca	(a,a)
acb	(a,a)
acc	(a,a)
baa	(b,a)
bab	(b,a)
bac	(b,a)
bba	(b,a)
bbb	(b,b)
bbc	(b,a)
bca	(b,a)
bcb	(b,a)
bcc	(b,a)
caa	(c,c)
cab	(c,c)
cac	(c,c)
cba	(c,c)
cbb	(c,c)
cbc	(c,c)
cca	(c,c)
ccb	(c,c)
ccc	(c,c)

Table 2.2'

{ }		{ }		{ }		{ }		{ }		{ }	
aaa	(a,a)	baa	(b,a)	caa	(a,a)	daa	(d,d)	eaa	(b,a)	aaa	(a,a)
aab	(c,c)	bab	(e,e)	cab	(c,c)	dab	(f,f)	eab	(e,e)	aab	(c,c)
aac	(c,c)	bac	(e,e)	cac	(c,c)	dac	(f,f)	eac	(e,e)	aac	(c,c)
aad	(a,a)	bad	(b,a)	cad	(a,a)	dad	(d,d)	ead	(b,a)	aad	(a,a)
aae	(c,c)	bae	(e,e)	cae	(c,c)	dae	(f,f)	eae	(e,e)	aae	(c,c)
aaf	(c,c)	baf	(e,e)	caf	(c,c)	daf	(f,f)	eaf	(e,e)	aaf	(c,c)
aba	(d,d)	bba	(b,a)	cba	(a,a)	dba	(d,d)	eba	(b,a)	aba	(d,d)
abb	(a,b)	bbb	(b,b)	cbb	(c,c)	dbb	(f,f)	ebb	(e,e)	abb	(a,b)
abc	(f,f)	bbc	(e,e)	cbc	(c,c)	dbc	(f,f)	ebc	(e,e)	abc	(f,f)
abd	(d,d)	bbd	(b,a)	cbd	(a,a)	dbd	(d,d)	ebd	(b,a)	abd	(d,d)
abe	(f,f)	bbe	(e,e)	cbe	(c,c)	dbe	(f,f)	ebe	(e,e)	abe	(f,f)
abf	(f,f)	bbf	(e,e)	cbf	(c,c)	dbf	(f,f)	ebf	(e,e)	abf	(f,f)
aca	(a,a)	bca	(b,a)	cca	(a,a)	dca	(d,d)	eca	(b,a)	aca	(a,a)
acb	(c,c)	bcb	(e,e)	ccb	(c,c)	dcb	(f,f)	ecb	(e,e)	acb	(c,c)
acc	(c,c)	bcc	(e,e)	ccc	(c,c)	dcc	(f,f)	ecc	(e,e)	acc	(c,c)
acd	(a,a)	bcd	(b,d)	ccd	(a,a)	dcd	(d,d)	ecd	(b,a)	acd	(a,a)
ace	(c,c)	bce	(e,e)	cce	(c,c)	dce	(f,f)	ece	(e,e)	ace	(c,c)
acf	(c,c)	bcf	(e,e)	ccf	(c,c)	dcf	(f,f)	ecf	(e,e)	acf	(c,c)
ada	(a,a)	bda	(b,a)	cda	(a,a)	dda	(d,d)	eda	(b,a)	ada	(a,a)
adb	(c,c)	bdb	(e,e)	cdb	(c,c)	ddb	(f,f)	edb	(e,e)	adb	(c,c)
adc	(c,c)	bdc	(e,e)	cdc	(c,c)	ddc	(f,f)	edc	(e,e)	adc	(c,c)
add	(a,a)	bdd	(b,a)	cdd	(a,a)	ddd	(d,d)	edd	(b,a)	add	(a,a)
ade	(c,c)	bde	(e,e)	cde	(c,c)	dde	(f,f)	ede	(e,e)	ade	(c,c)
adf	(c,c)	bdf	(e,e)	cdf	(c,c)	ddf	(f,f)	edf	(e,e)	adf	(c,c)
aea	(d,d)	bea	(b,a)	cea	(a,a)	dea	(d,d)	eea	(b,a)	aea	(d,d)
aeb	(f,f)	beb	(e,e)	ceb	(c,c)	deb	(f,f)	eeb	(e,e)	aeb	(f,f)
aec	(f,f)	bec	(e,e)	cec	(c,c)	dec	(f,f)	eec	(e,e)	aec	(f,f)
aed	(d,d)	bed	(b,a)	ced	(a,a)	ded	(d,d)	eed	(b,a)	aed	(d,d)
aee	(f,f)	bee	(e,e)	cee	(c,c)	dee	(f,f)	eee	(e,e)	aee	(f,f)
aef	(f,f)	bef	(e,e)	cef	(c,c)	def	(f,f)	eef	(e,e)	aef	(f,f)
afa	(a,a)	bfa	(b,a)	cfa	(a,a)	dfa	(d,d)	efa	(b,a)	afa	(a,a)
afb	(c,c)	bfb	(e,e)	cfb	(c,c)	dfb	(f,f)	efb	(e,e)	afb	(c,c)
afc	(c,c)	bfc	(e,e)	cfc	(c,c)	dfc	(f,f)	efc	(e,e)	afc	(c,c)
afd	(a,a)	bfd	(b,a)	cfd	(a,a)	dfd	(d,d)	efd	(b,a)	afd	(a,a)
afe	(c,c)	bfe	(e,e)	cfe	(c,c)	dfe	(f,f)	efe	(e,e)	afe	(c,c)
aff	(c,c)	bff	(e,e)	cff	(c,c)	dff	(f,f)	eff	(e,e)	aff	(c,c)

Table 3.2"

6. Constructing the (3,2)-semigroup automaton

The (3,2)-automaton is given by the map $f : S \times B^2 \to S \times B$. Since we have extendet the alphabet B to B', we extend the (3,2)-automaton by introducing new pairs $(u, v) \in (B' \times B') \setminus (B \times B)$, and by defining $f'(s, x, y)$.

The partially extended (3,2)-semigroup automata for the (3,2)-automata from Examples 2 and 3 are given by the Tables 2.1" and 3.1".

f''	(a,a)	(a,b)	(a,c)	(b,a)	(b,b)	(b,c)	(c,a)	(c,b)	(c,c)
s_0	(s_0,a)	(s_0,b)		(s_0,a)	(s_0,a)				(s_0,a)
s_1	(s_0,a)	(s_2,b)		(s_2,a)	(s_2,a)				(s_2,a)
s_2	(s_2,a)	(s_2,a)		(s_2,a)	(s_2,b)				(s_2,a)

Table 2.1"

f''	(a,a)	(a,b)	(a,c)	(a,d)	(a,e)	(a,f)	(b,a)	(b,b)	(b,c)
s_0	(s_0,a)	(s_0,b)					(s_0,a)	(s_0,b)	
s_1	(s_0,a)	(s_2,b)					(s_2,a)	(s_2,a)	
s_2	(s_2,a)	(s_2,a)					(s_2,a)	(s_2,b)	

f''	(b,d)	(b,e)	(b,f)	(c,a)	(c,b)	(c,c)	(c,d)	(c,e)	(c,f)
s_0						(s_0,b)			
s_1						(s_0,b)			
s_2						(s_2,a)			

f''	(d,a)	(d,b)	(d,c)	(d,d)	(d,e)	(d,f)	(e,a)	(e,b)	(e,c)
s_0				(s_0,a)					
s_1				(s_2,a)					
s_2				(s_2,a)					

f''	(e,d)	(e,e)	(e,f)	(f,a)	(f,b)	(f,c)	(f,d)	(f,e)	(f,f)
s_0		(s_0,b)							(s_0,b)
s_1		(s_2,a)							(s_2,a)
s_2		(s_2,a)							(s_2,a)

Table 3.1"

While generating the (3,2)-semigroup, $(B', \{\})$, we generate also the (3,2)-semigroup automaton (S, B', f'), for the given (3,2)-automaton (S, B, f), where $f' : S \times B'^2 \to S \times B$, such that

$$f'(s_i, x, y) = f(s_i, x, y) \text{ for } x, y \in B.$$

After we generate the (3,2)-semigroup $(B', \{\})$, it is possible not all $f'(s_i, x, y)$ to be defined. Using the fact that $\{xyz\} = (u_1, v_1)$ and $\{zxy\} = (u_2, v_2)$ for some $u_1, u_2, v_1, v_2 \in B'$, we define $f'(s_i, x, y)$ according to the condition:

$$\begin{aligned} (iii_1) \quad & f'(s_i, u_1, v_1) = f'(s_i, \{xyz\}) = f'(f'(s_i, x, y), z) \\ & f'(s_i, u_2, v_2) = f'(s_i, \{zxy\}) = f'(f'(s_i, z, x), y), \end{aligned}$$

where $f'(s_i, u_1, v_1)$ and $f'(s_i, u_2, v_2)$ are already defined.

We continue in this way, using the condition (iii_1), until all $f'(s_i, x, y)$, are defined, i.e. until the (3,2)-semigroup automaton is not defined.

The (3,2)-semigroup automata for the (3,2)-automata from Examples 2 and 3 are given by Tables 2.2" and 3.2".

f''	(a,a)	(a,b)	(a,c)	(b,a)	(b,b)	(b,c)	(c,a)	(c,b)	(c,c)
s_0	(s_0, a)	(s_0, b)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
s_1	(s_0, a)	(s_2, b)	(s_0, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)

Table 2.2"

f''	(a,a)	(a,b)	(a,c)	(a,d)	(a,e)	(a,f)	(b,a)	(b,b)	(b,c)
s_0	(s_0, a)	(s_0, a)	(s_0, b)	(s_0, a)	(s_0, b)	(s_0, b)	(s_0, a)	(s_0, b)	(s_0, b)
s_1	(s_0, a)	(s_2, b)	(s_0, a)	(s_0, a)	(s_2, a)	(s_0, a)	(s_2, a)	(s_2, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, b)	(s_2, a)

f''	(b,d)	(b,e)	(b,f)	(c,a)	(c,b)	(c,c)	(c,d)	(c,e)	(c,f)
s_0	(s_0, a)	(s_0, b)	(s_0, b)	(s_0, a)	(s_0, a)	(s_0, b)	(s_0, a)	(s_0, a)	(s_0, a)
s_1	(s_2, a)	(s_2, a)	(s_2, a)	(s_0, a)	(s_0, a)	(s_0, b)	(s_0, a)	(s_0, a)	(s_0, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)

f''	(d,a)	(d,b)	(d,c)	(d,d)	(d,e)	(d,f)	(e,a)	(e,b)	(e,c)
s_0	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)
s_1	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)

f''	(e,d)	(e,e)	(e,f)	(f,a)	(f,b)	(f,c)	(f,d)	(f,e)	(f,f)
s_0	(s_0, a)	(s_0, b)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, a)	(s_0, b)
s_1	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)
s_2	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)	(s_2, a)

Table 3.2"

7. Algorithm for generating (3,2)-semigroup automata

Phaze 1:

Step 1.1 Extend the table of the given (3,2)-automaton with new columns $\{xyz\}$ for $x, y, z \in B$ and fill it up using the rule:

$$f(f(s, x, y), z) = f(s, \{xyz\}).$$

Step 1.1.1 If for each triple $\{xyz\}$ there is at least on pair (u, v), for $u, v \in B$, such that their columns coincide, i.e. the condition

$$f(f(s, x, y), z) = f(s, u, v), \text{ for every } s \in S \qquad (*)$$

is satisfied, then at least one (3,2)-groupoid is generated by the given (3,2)-automaton.

Step 1.1.2 If there is a triple $\{xyz\}$ without a corresponding pair (u, v), then the (3,2)-automaton does not generate a (3,2)-gruopoid over the set B.

Step 1.2 Extend the table with new rows (s_i, x), for every $s_i \in S$ and every $x \in B$. Fill up the extended table using the rules:

$$\begin{aligned} f((s_i, a), (u, v)) &= f(f(s_i, a, u), v) \\ f((s_i, a), \{xyz\}) &= f(f(f(s_i, a, x), y), z) \end{aligned} \qquad (4)$$

and again if some of the extended columns for the triples $\{xyz\}$ for $x, y, z \in B$ coincide with some of the extended columns for the pairs (u, v), i.e. if the condition

$$f((s_i, a), (u, v)) = f((s_i, a), \{xyz\}) \text{ for every } s_i \in S \text{ and every } x \in B, \qquad (**)$$

is satisfied.

Step 1.2.1 If for each triple $\{xyz\}$ there is a pair (u, v), for $u, v \in B$, such that their columns coincide, i.e. the condition (**) is satisfied, then at least one (3,2)-semigroup over B is generated by the given (3,2)-automaton. The extended table shows how the (3,2)-semigroup is defined, and how the (3,2)-semigroup automaton is defined. Here is the end of the algorithm.

Step 1.2.2 If there is a triple $\{xyz\}$ without a corresponding pair (u, v), as in Step 1.2.1, then the (3,2)-automaton does not generate a (3,2)-semigroup over the set B. It is possible Step 1.1.1 to be satisfied, but Step 1.2.1 not to be satisfied.

Step 1.3 For each triple $\{xyz\}$ which does not satisfy Step 1.1.1 or Step 1.2.1, we introduce new letter, say p, extend the alphabet B to a new alphabet B', and define $\{xyz\} = (p, p)$. Next, we extend the already extended table with new columns $\{xyzt\}$ for $x, y, z, t \in B$, and fill it up again using the rule (3). Again we compare the new columns with the old ones.

Step 1.3.1 If each of the new columns coincide with some of the old ones, then we go to Phase 2.

Step 1.3.2 If some of the new column does not coincide with some of the old columns, we introduce new letter, say q, extend the alphabet, and to this column we assign the pair (q, q).

We repeat the Step 1.3, finitely many time, with new columns $\{xyztu\}$, $\{xyztuv\}$ and so on, for $x, y, z, t, u, v \in B$, until all the new columns coincide with some of the previous ones. At the end we have new alphabet, denoted by B'.

Then we go to Phase 2.

Phase 2:

Step 2.1 The extended table from Phase 1, gives the definition of the (3,2)-operation for the triples $\{xyz\}$ for $x, y, z \in B$, and $\{xtt\}$ and $\{ttx\}$ for $x \in B$ and $t \in B' \setminus B$.

Step 2.2 If for some $x \in B' \setminus B$, $\{xxx\}$ is not already defined, we define $\{xxx\} = (x, x)$, and then we define all of its consequences, according to the rules:

$$(ii_1) \quad \begin{aligned} \{\{xxx\}y\} &= \{x\{xxy\}\} \\ \{y\{xxx\}\} &= \{\{yxx\}x\} \text{ for } y \in B'; \end{aligned}$$

$$(ii_2) \quad f(s_i, \{xxx\}) = f(f(s_i, x, x), x) \text{ for every } s_i \in S.$$

Step 2.3 We define $\{abx\}, \{axb\}, \{xab\}$ for $a, b \in B$ and $x \in B' \setminus B$ according to the rules:

$$(iii_1) \quad \begin{aligned} \{y\{abx\}\} &= \{\{yab\}x\} \\ \{y\{axb\}\} &= \{\{yax\}b\} \end{aligned}$$

$$\{y\{xab\}\} = \{\{yxa\}b\}, \quad \text{for } y \in B';$$

$$(iii_2) \quad \begin{aligned} &f(s_i, \{abx\}) = f(f(s_i, a, b)x) \\ &f(s_i, \{axb\}) = f(f(s_i, a, x)b) \\ &f(s_i, \{xab\}) = f(f(s_i, x, a)b), \quad \text{for every } s_i \in S; \end{aligned}$$

(iii_3) if there is more than one (u, v) satisfying the conditions (iii_1) and (iii_2), then we choose the one, such that it is equal to some already defined $\{xyz\}$ in the previous steps.

After this, define all the consequences for the (3,2)-operation $\{\}$ and the extension of the transition function f'.

Extend the alphabet B with the letter $x \in B' \setminus B$, i.e. set $B = B \cup \{x\}$, and repeat the Step 2.3 for the new B. Continue this, until $B = B'$, i.e. until all $\{xyz\}$ for $x, y, z \in B'$ are defined.

Step 2.4 After the (3,2)-semigroup $(B', \{\})$ is defined, it is possible the transition function not to be defined, i.e. some $f'(s_i, x, y)$ not to be defined. Using the fact that $\{xyz\} = (u_1, v_1)$ and $\{zxy\} = (u_2, v_2)$ for some $u_1, u_2, v_1, v_2 \in B'$, we define $f'(s_i, x, y)$ according to the condition:

$$(iii_1) \quad \begin{aligned} f'(s_i, u_1, v_1) &= f'(s_i, \{xyz\}) = f'(f'(s_i, x, y), z) \\ f'(s_i, u_2, v_2) &= f'(s_i, \{zxy\}) = f'(f'(s_i, z, x), y), \end{aligned}$$

where $f'(s_i, u_1, v_1)$ and $f'(s_i, u_2, v_2)$ are already defined.

We repeat Step 2.4 until all $f'(s_i, x, y)$ are defined, i.e. the (3,2)-semigroup automaton is defined.

We will mention here that we have a computer program for the above algorithm, which generates (3,2)-semigroups and (3,2)-semigroup automata for given (3,2)-automata.

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