CONSEQUENCES OF A THEOREM OF H. MARGOLIS

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1. *Introduction*

We adopt the notation and conventions from [**1**]. So we write Y_* and Y^* for the homology and cohomology theories respectively, associated to a spectrum Y. The sphere spectrum is denoted by S, the Eilenberg MacLane spectrum for the abelian group G by HG. The Steenrod algebra for the prime p is $A_p = HZ_p{}^*(HZ_p)$ and its dual $(A_p)_* = HZ_{p*}(HZ_p)$, where Z_p is the group of integers modulo p. When there is no danger of confusion, the p in A_p and $(A_p)_*$ will be omitted.

Let Ch be the functor from the category of locally compact Hausdorff abelian groups to itself defined for any such group G by $Ch\,G = \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{R}/\mathbf{Z})$ where $\mathbf{R}/\mathbf{Z}$ is the group of reals modulo the integers with its usual topology, and the right-hand side is the internal Hom-functor. Ch is an exact functor, sends compact groups to discrete ones and vice versa. Therefore, for any spectrum Y, the composite $Ch \circ Y_*$ is a cohomology theory with values in the category of compact Hausdorff abelian groups. Let Y^c denote the spectrum corresponding to $U \circ Ch \circ Y_*$ where U is the functor that forgets the topology. E. H. Brown [3] studied the connection between Y and Y^c for certain classes of Y. For example, he showed that

$$\pi_i(Y^c) \cong U \circ Ch\pi_{-i}(Y)$$

for any spectrum Y. Furthermore, if Y and Z are connective spectra with $\pi_i(Y)$ and $\pi_i(Z)$ finite for each i, there is a natural isomorphism $[Y, Z]_n \xrightarrow{\cong} [Z^c, Y^c]_n$ for each $n \in \mathbf{Z}$, where $[Y, Z]_n$ denotes the set of homotopy classes of maps from Y to Z of degree n. He also announced the following

THEOREM 1. *If Y is a connective spectrum with $\pi_i(Y)$ finite for each i, then the map*

$$h : [HZ_p, Y] \to \mathrm{Hom}_{A_p}(HZ_p{}^*(Y), A_p)$$

given for each $f \in [HZ_p, Y]$ by $h(f) = f^*$, the induced map in cohomology, is an isomorphism.

We somewhat extend these results and thus compute $S^*(HZ_p)$ and $HZ_p{}^*(S^c)$.

I wish to express my gratitude to Professor J. F. Adams for much help and encouragement in the preparation of my Doctoral thesis of which this paper is an appendix.

2. *On Theorem* 1

We begin by proving the dual of Theorem 1.

THEOREM 1′. *If Y is a connective spectrum with $\pi_i(Y)$ finite for each i, then*

$$h : [HZ_p, Y] \to \mathrm{Hom}_{A_*}(A_*, HZ_{p*}(Y))$$

given for each $f \in [HZ_p, Y]$ by $h(f) = f_*$, the induced map in homology, is an isomorphism.

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Proof. The Steenrod algebra is injective as a (left or right) module over itself [4]. Note that A_p is locally finite. If M^* is a locally finite graded A_p-module, its dual M_* is a locally finite graded $(A_p)_*$-comodule. In this case we have

$$\mathrm{Hom}_{A_p}(M^*, A_p) \cong \mathrm{Hom}_{(A_p)_*}((A_p)_*, M_*).$$

Since $\mathrm{Hom}_{A_p}(-, A_p)$ is exact, it follows that $\mathrm{Hom}_{(A_p)_*}((A_p)_*, -)$ is exact on the category of locally finite $(A_p)_*$-comodules.

Consider the Adams spectral sequence [1; Theorem 15.1] for $E = HZ_p$, $X = HZ_p$ and $Y = Y$. By the above, the E_2-term $\mathrm{Ext}^{s*}_{A_*}(A_*, HZ_{p*}(Y))$ is zero for $s > 0$. Indeed, since each homotopy group of Y is finite, $HZ_{p*}(Y)$ is locally finite.

Since Y is connective and HZ_p has all the properties needed for the convergence of the sequence, it does so towards

$$[HZ_p, Y]^{HZ_p} \cong [HZ_p, Y^{HZ_p}]$$

where Y^{HZ_p} denotes the HZ_p completion of Y [1]. We have to show that

$$[HZ_p, Y^{HZ_p}] \cong [HZ_p, Y].$$

But since the homotopy groups of Y are finite, the map

$$\alpha : Y \to \prod_q Y^{HZ_q}$$

where q runs over the set of primes, is an equivalence. Indeed, let MI_q denote the Moore spectrum for the q-adic integers I_q. Then

$$Y^{HZ_q} \cong MI_q \wedge Y = YI_q$$

and

$$\pi_*(YI_q) \cong \pi_*(Y) \otimes I_q.$$

The finiteness of $\pi_i(Y)$ implies that

(a) $\pi_*(Y) \otimes I_q \cong {}_q\pi_*(Y)$ where ${}_qG$ denotes the q-primary component of the group G, and

(b) $\pi_i(Y)$ is a direct product over q of its q-primary components.

So the induced homomorphism by α in homotopy is an isomorphism and therefore α is an equivalence.

Now

$$\left[HZ_p, \prod_q Y^{HZ_q}\right] \cong \prod_q [HZ_p, Y^{HZ_q}] \cong [HZ_p, Y^{HZ_p}],$$

where the second isomorphism follows from $[HZ_p, Y^{HZ_q}] \cong 0$ for $p \neq q$.

Remark 1. Since for Y satisfying the conditions of Theorem 1′ $HZ_{p*}(Y)$ and $HZ_p^*(Y)$ are locally finite and dual, we get

$$\mathrm{Hom}_A(HZ_p^*(Y), A) \cong \mathrm{Hom}_{A_*}(A_*, HZ_{p*}(Y))$$

and hence Theorem 1.

Remark 2. The referee has pointed out that Theorem 1 (at least for Y with $\pi_*(Y)$ finite and $p = 2$) was observed by Margolis to be a consequence of his result that A_2 is injective [2] (see (1) and (2) below) and that an alternative proof of Theorem 1 can be obtained by the following argument:

(1) The theorem is true for $Y = HZ_q$, q prime.

(2) If $F \to E \to B$ is a fibration of spectra and the theorem is true for F and B it is true for E (applying $[HZ_p, -] \to \mathrm{Hom}_A(HZ_p{}^*(-), A)$ gives a ladder of exact sequences). Hence Theorem 1 is true for Y with $\pi_*(Y)$ finite.

(3) A Postnikov system for Y and a limit argument give the general result.

THEOREM 2. *The conclusion of Theorem 1 holds for any connective spectrum Y such that each $\pi_i(Y)$ is finitely generated.*

The proof uses a trick that I first heard from Mr. D. Baird.

Consider the cofibering $Y \xrightarrow{p} Y \to Y \wedge MZ_p$ where MZ_p denotes the Moore spectrum for the group Z_p. Now $Y \wedge MZ_p$ satisfies the conditions of Theorem 1. Therefore in the diagram

$$\begin{array}{ccccccc} \cdots \to [HZ_p, Y] & \xrightarrow{p} & [HZ_p, Y] & \to & [HZ_p, Y \wedge MZ_p] & \to \cdots \\ \downarrow h_1 & & \downarrow h_1 & & \downarrow h & \\ \cdots \to \mathrm{Hom}_A(HZ_p{}^*(Y), A) & \xrightarrow{p} & \mathrm{Hom}_A(HZ_p{}^*(Y), A) & \to & \mathrm{Hom}_A(HZ_p{}^*(Y \wedge MZ_p), A) & \to \cdots \end{array}$$

with exact rows, the map h is an isomorphism. Since every element in $[HZ_p, Y]$ and $HZ_p{}^*(Y)$ is of order p (that follows from the fact that multiplication by p on them can be thought to be induced from $HZ_p \xrightarrow{p} HZ_p$ which is trivial) the maps " multiplication by p " in the diagram above are zero. Then an easy diagram-chasing argument shows that h_1 is both a monomorphism and an epimorphism.

COROLLARY 1. $S^*(HZ_p) \cong 0$.

This follows from $\mathrm{Hom}_A(Z_p, A) \cong 0$.

3. *The isomorphism* $[Y, Z] \cong [Z^c, Y^c]$

We now turn to the map $[Y, Z]_n \to [Z^c, Y^c]_n$ which is defined in the following way. A homotopy class of maps $Y \to Z$ of degree n induces a homology operation $Y_* \to Z_{*+n}$. But the set $(Y_*, Z_*)_n$ of homology operations of degree n is isomorphic to the set of continuous cohomology operations of degree n from $Ch \circ Z_*$ to $Ch \circ Y_{*+n}$. Denote that set by $(Z^{c*}, Y^{c*})_n{}^{cts}$. Obviously, $(Z^{c*}, Y^{c*})_n{}^{cts} \subset [Z^c, Y^c]_n$.

THEOREM 3. *The map $[Y, Z]_n \to [Z^c, Y^c]_n$ defined above is an isomorphism in either of the following cases:*

(1) *each homotopy group of Z is finite,*

(2) *each homotopy group of Z is finitely generated and there exists an integer m such that the map $Y \xrightarrow{m} Y$ is trivial.*

Proof. (1) First we show that in this case $[Y, Z]_n \cong (Y_*, Z_*)_n$. The inverse r to the obvious map $[Y, Z]_n \to (Y_*, Z_*)_n$ is constructed in the following way. Given an element $\alpha \in (Y_*, Z_*)_n$, by Spanier Whitehead duality, it induces a cohomology operation $\alpha_X^* : Y^*(X) \to Z^{*+n}(X)$ on finite spectra X. Since $\pi_i(Z)$ is finite, there is an isomorphism [5; p. 37]

$$\varprojlim_{Y_f} [Y_f, Z]_n \cong [Y, Z]_n$$

where Y_f ranges over the finite subspectra of Y. Therefore to the element $\{i_f\} \in \varprojlim_{Y_f} [Y_f, Y]$ where i_f is the class of the inclusion of Y_f into Y, corresponds under α^* and the above isomorphism a morphism $f \in [Y, Z]_n$ which is the image of α under r.

Further, $(Z^{c*}, Y^{c*})_n^{cts}$ is then isomorphic to $[Z^c, Y^c]_n$. For, given a cohomology operation $\theta \in [Z^c, Y^c]_n$, let X be a finite spectrum. Then $Ch \circ Z_*(X)$ is finite and therefore discrete, so θ_X is continuous. But for any spectrum X, we have

$$Ch \circ Z_*(X) \cong \varprojlim_{X_f} Ch \circ Z_*(X_f)$$

and

$$Ch \circ Y_*(X) = \varprojlim_{X_f} Ch \circ Y_*(X_f)$$

where X_f runs over the finite subspectra of X. So the continuity of θ on finite spectra implies its continuity.

(2) Consider the cofibering $Z \xrightarrow{m} Z \to Z \wedge MZ_m$. Then in the diagram

$$\begin{array}{ccccccc}
\ldots \to & [Y, Z]_n & \xrightarrow{m} & [Y, Z]_n & \to & [Y, Z \wedge MZ_m]_n & \to \ldots \\
& \downarrow k_1 & & \downarrow k_1 & & \downarrow k & \\
\ldots \to & (Y_*, Z_*)_n & \to & (Y_*, Z_*)_n & \to & (Y_*, (Z \wedge MZ_m)_*)_n & \to \ldots \\
& \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
\ldots \to & (Z^{c*}, Y^{c*})_n^{cts} & \to & (Z^{c*}, Y^{c*})_n^{cts} & \to & ((Z \wedge MZ_m)^{c*}, Y^{c*})_n & \to \ldots \\
& \downarrow i_1 & & \downarrow i_1 & & \downarrow i & \\
\ldots \to & [Z^c, Y^c]_n & \xrightarrow{m} & [Z^c, Y^c]_n & \to & [(Z \wedge MZ_m)^c, Y^c]_n & \to \ldots
\end{array}$$

the top and bottom rows are exact, m is zero and $i \circ k$ is an isomorphism by (1). Therefore the same diagram-chasing argument used in the proof of Theorem 2 shows that $i_1 \circ k_1$ is an isomorphism.

COROLLARY 2. $HZ_p{}^*(S^c) \cong 0$.

Remark 3. The spectrum S^c is interesting, for Y^c is equivalent to the function spectrum $F(Y, S^c)$ for any spectrum Y.

References

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