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TOPOLOGICAL OBJECTS IN HOMOTOPY THEORY

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1. Introduction

A topological object in a category C consists of an object X in that category together with a topology on the sets $\mathrm{Mor}(Y, X)$ for each $Y \in \mathrm{Ob}\, C$ so that $\mathrm{Mor}(\ , X)$ thus defines a functor from C to the category of topological spaces and continuous maps.

There are at least two cases of interest in homotopy theory of such topological objects.

1. For a space X, let $\hat{X}$ denote Sullivan's profinite completion of it [9]. Then for each space Y, the set $[Y, \hat{X}]$ of homotopy classes of maps $Y \to \hat{X}$ has a compact Hausdorff totally disconnected topology which is natural with respect to Y.

2. (E. H. Brown) Let H be a generalized homology theory in the category of CW spectra [1]. Let Ch denote Pontrjagin's character functor. Then the composite $Ch \circ H$ is a generalized cohomology theory with values in the category of compact Hausdorff abelian groups and continuous homomorphisms. Let H^c be the spectrum representing $U \circ Ch \circ H$, where U is the functor that forgets the topology. Then for any spectrum Y, $[Y, H^c]$ may be given a natural compact Hausdorff topology, namely that of $Ch \circ H_*^c(Y)$.

Let C be the category of CW complexes and homotopy classes of maps. Let X, $Y \in \mathrm{Ob}\, C$ and let $[Y, X]$ denote $\mathrm{Mor}_C(Y, X)$. In Section 2 we propose a definition for the representability of a functor $[\ , X] : C \to$ category of compact Hausdorff spaces and continuous maps by giving X some extra structure allowing one to infer from it the topology on $[Y, X]$ for each Y. The definition is the following; if K is a Kan simplicial compact Hausdorff space, then for each simplicial set L the set (L, K) of simplicial maps $L \to K$ has a natural compact Hausdorff topology and so has the set $[L, K]$ of homotopy classes of such maps. Moreover the topology on $[L, K]$ depends only on the homotopy type of the realization of L. Let $|K|$ denote the realization of K considered as a simplicial set and let $\mathrm{Sin}\, Y$ denote the singular complex of Y. By defining a topology on $[Y, |K|]$ so that the natural composition

$$[Y, |K|] \to [|\mathrm{Sin}\, Y|, |K|] \overset{\cong}{\to} [\mathrm{Sin}\, Y, K]$$

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is a homeomorphism, K defines a compact Hausdorff object in C. Then we say that $[\ ,\ X]$ is representable if there is a Kan simplicial compact Hausdorff space K such that there is a natural equivalence (of topological space valued functors) between $[\ ,\ |K|]$ and $[\ ,\ X]$.

Section 3 shows that a Kan simplicial compact Hausdorff L-prespectrum (the simplicial object corresponding to a topological Ω-spectrum but with the additional structure described above) defines a generalized cohomology theory with values in the category of compact Hausdorff abelian groups and continuous homomorphisms.

In Section 4 the representability of such a theory is proved.

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2. Topological objects

In this section we give the definition of a topological object in a category and show how simplicial spaces define various such objects: in the category of simplicial sets, the homotopy category of simplicial sets, etc.

2.1. DEFINITION. *An object B in a category C together with a topology on the sets* Mor $(A,\ B)$ *for each* $A \in \operatorname{Ob} C$ *is said to be a topological object in C if* Mor $(\ ,\ B)$ *thus defines a functor from C to the category of topological spaces and continuous maps.*

2.2. EXAMPLE. Let $K = \{K_n,\ n \geq 0\}$ be a simplicial topological space and let $L = \{L_n,\ n \geq 0\}$ be a simplicial set. Let $(L,\ K)$ be the set of maps in the category of simplicial sets from L to the simplicial set underlying K. Then $(L,\ K)$ may be considered as a subset of the product space

$$\prod_{i \geq 0} (K_i)^{L_i}$$

We define the A-topology on $(L,\ K)$ to be that which it inherits as a subspace of this product space.

Note. We shall sometimes write Mor $(A,\ B)_X$ to mean that Mor $(A,\ B)$ is considered with the X-topology.

2.3. EXAMPLE. Let K be a Kan simplicial space. Let L be a simplicial set. Let $[L,\ K]$ be the set of homotopy classes of simplicial maps from L to K. It is a quotient set of $(L,\ K)$. We define the B-topology on $[L,\ K]$ to be that which it inherits as a quotient space of $(L,\ K)_A$.

2.4. LEMMA. *If K is a Kan simplicial compact Hausdorff space, L a simplicial set, then the topology on $[L,\ K]$ defined above is compact Hausdorff.*

Proof. Compactness is easy to check. Indeed, (L, K) is a closed subspace of a product of compact spaces, and $[L, K]$ is a quotient space of (L, K).

In order to prove Hausdorffness it is enough to show that the map

$$\alpha : (L, K) \to [L, K]$$

is closed. For under the conditions of the lemma (L, K) is compact and Hausdorff and therefore normal. So $[L, K]$ as its image under a closed map would be normal. But every point in (L, K) is closed, so $[L, K]$ would also be T_1 and therefore Hausdorff.

Now to prove that α is a closed map, first note that the operator d_1 in the simplicial space K^L

$$d_1 : (K^L)_1 \to (K^L)_0$$

is a closed map. Indeed, $(K^L)_1$ is compact, d_1 is closed. But if d_1 is a closed map, so is α. For all we have to check is that the set G of all maps $L \to K$ each of which is homotopic to some map in an arbitrarily given closed set $F \subset (K^L)_0 = (L, K)$ is closed. Now G is given by $G = d_1 d_0^{-1} F$ ([7], Proposition 6.2), and since d_0 is continuous and d_1 is closed, the result follows.

For a simplicial set K, let $|K|$ denote the realisation of K. For a topological space X, let $\operatorname{Sin} X$ denote the singular complex of X. It is well-known that $|\ |$ and Sin are adjoint functors. Furthermore, if K is Kan, $\operatorname{Sin}|K|$ has the same homotopy type as K; if X is a CW complex $|\operatorname{Sin} X|$ is homotopy equivalent to X. Therefore, for X a *CW* complex and K a Kan simplicial set, there is a 1—1 correspondence

$$[X, |K|] \cong [\operatorname{Sin} X, K]$$

where $[X, |K|]$ denotes the set of homotopy classes of maps from X to $|K|$.

2.5. DEFINITION. *Let K be a Kan simplicial space and let X be a CW complex. Let $|K|$ denote the realization of the simplicial set underlying K. We define the C-topology on $[X, |K|]$ to be that which corresponds under the 1—1 correspondence defined above to the B-topology on* $[\operatorname{Sin} X, K]$.

2.6. EXAMPLE. Suppose K is a Kan simplicial compact Hausdorff space; let L be a simplicial set, and take $X = |L|$ in the above. Then we obtain the C-topology on $[|L|, |K|]$.

2.7. LEMMA. *Under the 1—1 correspondence $[L, K] \to [|L|, |K|]$ given by $f \mapsto |f|$ for each $f \in [L, K]$, the B-topology on $[L, K]$ corresponds to the C-topology on $[|L|, |K|]$.*

Proof. The canonical map $L \to \mathrm{Sin}\,|L|$ induces a continuous 1—1 map from $[L, K]$ to $[\mathrm{Sin}\,|L|, K]$. Both spaces being compact Hausdorff, this map is a homeomorphism. Now the diagram

$$\begin{array}{ccc} [L, K] & \xrightarrow{f \mapsto |f|} & [|L|, |K|] \\ & \searrow \cong \quad \cong \nearrow & \\ & [\mathrm{Sin}\,|L|, K] & \end{array}$$

where the right hand side isomorphism defines the C-topology on $[|L|, |K|]$, implies the lemma.

2.8. DEFINITION. *Let K be a simplicial space and let $|K|_T$ denote the realization of K remembering the topology on the sets K_n. Let X be a CW complex. Define the D-topology on $[X, |K|]$ in the following way. It is a quotient space of the space of continuous maps $(X, |K|)$ considered as a subspace of $(X, |K|_T)$ with the compact open topology. It is easy to see that this defines a topological object in the homotopy category of CW complexes.*

Given a simplicial set L, there is a map

$$R : [L, K]_S \to [|L|, |K|]_D$$

defined by $[\varphi] \mapsto [|\varphi|]$ for any $\varphi \in (L, K)$.

2.9. PROPOSITION. *Let K be a Kan simplicial compact Hausdorff space, L a simplicial set. Then the map R defined above is continuous.*

Proof. The diagram

$$\begin{array}{ccc} (L, K) & \xrightarrow{\tilde{R}} & (|L|, |K|) \\ \downarrow & & \downarrow \\ [L, K] & \xrightarrow{R} & [|L|, |K|] \end{array}$$

where the bottom spaces are quotient spaces of the top ones, implies that it is enough to check that $\tilde{R}$ is continuous.

Take an elementary open set (C, O) in $(|L|, |K|)$ given by $(C, O) = \{f : |L| \to |K| : f(C) \subset O,\ C$ compact in $|L|$, O open in $|K|_T\}$.

Since C is compact there are a finite number of closed cells, say $\sigma_1, \ldots, \sigma_k$ of $|L|$ covering C. Denote $C_i = C \cap \sigma_i$. Let $s_1, \ldots, s_k$ be the nondegenerate simplices of L corresponding to $\sigma_1, \ldots, \sigma_k$. Let $\varphi : L \to K$ be such that $|\varphi| \in (C, O)$. We have to show that there is a neighbourhood M of φ in (L, K) such that for each $\psi \in M$, $|\psi| \in (C, O)$. Fix i. Each $x \in |\varphi|(C_i)$ is contained in a subset of O of the form

$$O_i^x \times N^x_{\varphi s_i}$$

where O_i^x is open in the space of the standard r-simplex $\Delta[r]$ and $N^x_{\varphi s_i}$ is an open neighbourhood of φs_i in K_r (here r denotes the dimension of s_i). The sets $O_i^x \times N^x_{\varphi s_i}$ cover $|\varphi|(C_i)$ when x ranges over all points of $|\varphi|(C_i)$. Since the latter is compact, there are finitely many points $x_1, \ldots, x_n \in |\varphi|(C_i)$ which determine a finite subcover. Let

$$N_{\varphi s_i} = \bigcap_{j=1}^{r} N^{x_j}_{s_i}.$$

Put $M = \{\psi : L \to K : \psi(s_i) \in N_{\varphi s_i},\ i = 1, \ldots, k\}$. The set M is open in (L, K) and $\varphi \in M$. Furthermore, the realization of each ψ in M is in (C, O) because for each $y \in C_i$ such that $|\psi|(y) = x$ we have

$$|\psi|(y) \in O_i^x \times N_{\varphi s_i} \subset O_i^x \times N^x_{\varphi s_i} \subset O.$$

2.10. COROLLARY. *If K is a Kan simplicial compact Hausdorff space, $[X, |K|]_D$ is compact.*

2.11. COROLLARY. *For any CW complex X, the identity function $[X, |K|]_C \to [X, |K|]_D$ is continuous.*

Proof. Let L be a simplicial set whose realization is homotopy equivalent to X. Now use the diagram

$$\begin{array}{ccc} [X, |K|]_C & & [X, |K|]_D \\ \downarrow\uparrow & & \downarrow\uparrow \\ [|L|, |K|]_C & \longrightarrow & [|L|, |K|]_D \end{array}$$

Proposition 2.9. and Lemma 2.7.

2.12. REMARK. Although the D-topology is in general coarser than the C-topology, they agree when K is an inverse limit of Kan simplicial finite sets.

2.13. DEFINITION. *A contravariant compact Hausdorff space valued functor F on the homotopy category of pointed CW complexes is said to be representable if there is a pointed Kan simplicial compact Hausdorff space K such that F is naturally equivalent to $[\ , |K|]_C$.*

2.14. EXAMPLE. Let $F = H^n(\ , G) = HG^n$ be the n-th cohomology functor [5] where G is a compact Hausdorff abelian group.

Then F is representable in the sense of Definition 2.13 and the representing object is the Kan simplicial compact Hausdorff space $EM(G, n)$ defined by

$$(EM(G, n))_q = Z^n(\Delta[q], G)$$

where $Z^n(\Delta[q], G)$ is the space of n-cocycles of the standard q-simplex with values in the compact Hausdorff abelian group G. This simplicial space is the classical Eilenberg-MacLane complex of type (G, n).

We now state two facts that will be needed in what follows.

2.15. *Fact*. The inverse limit over a small filtering category of non-empty compact Hausdorff spaces is non-empty.

This is proved by showing that the inverse limit $\varprojlim_J F_\alpha$ of a diagram of compact Hausdorff spaces F_α indexed by a small filtering category J is isomorphic to the inverse limit $\varprojlim_D I_\beta$ of compact Hausdorff spaces I_β indexed by a directed set D. Then the result is classical ([2], Proposition 8, 1.9.6).

2.16. *Fact*. A compact Hausdorff object Y in the homotopy category of CW complexes is uniquely determined by its values on the full subcategory generated by finite complexes.

Indeed,

$$[X, Y] \cong \varprojlim [X_f, Y]$$

where X_f runs over the finite subcomplexes of X [9].

This remark allows one to claim that a natural transformation between compact Hausdorff objects is continuous just by checking that fact on finite complexes.

3. The stable case

The aim of this section is to show that a Kan simplicial compact Hausdorff L-prespectrum defines a compact Hausdorff object in the stable homotopy category of pointed CW complexes.

All spaces (simplicial or topological) are going to be pointed. The base point will usually be denoted by $*$.

We need the definition of the looop space of a simplicial space. This, like some other that follow, is completely analogous to the one in the category of simplicial sets. So much so, in fact, that we shall often call upon a definition or result from [6] or [7] when what we really mean is the corresponding one in the category of simplicial spaces.

3.1. DEFINITION. *Let Y be a Kan simplicial space, where* $*$ *is the only vertex of* Y. *Define the path space of* Y *to be the simplicial space* PY, *by letting* $\lambda : Y_{n+1} \to (PY)_n$ *be a homeomorphism of spaces and by defining* $d_i = \lambda\, d_i\, \lambda^{-1}$, $s_i = \lambda\, s_i\, \lambda^{-1}$ $(0 \leq i \leq n)$. *Define a simplicial map* $p : PY \to Y$ *by* $p\,|\,(PY)_n = d_{n+1}\,\lambda^{-1}$ *and let the loop space* LY *of* Y *be the simplicial space* $LY = p^{-1}(*)$. *

It is easily verified that p is a Kan fibration of simplicial spaces and therefore PY and LY are Kan simplicial spaces ([7], Proposition 7.3 and 7.5). Moreover PY is contractible: as usual use the extra degeneracy map of Y.

3.2. DEFINITION. *Let* X *be a simplicial set with base point* $*$. *Define the suspension* EX *of* X *as in* [6], *i. e. let* P *be the simplicial set with exactly one non-degenerate simplex* Φ_n *for every* $n \geq 0$. *Let*

$$(EX)_n = \{(\sigma,\ \Phi) : \sigma \in X,\ \ \Phi \in P,\ \ \sigma \neq * \text{ and } \dim \sigma + \dim \Phi = n - 1\}$$

The face and degeneracy operators are given by

$$d_i(\sigma,\ \Phi) = \begin{cases} (d_i\,\sigma,\ \Phi) & 0 \leq i \leq p \\ (\sigma,\ d_{i-p-1}\,\Phi) & p < i \leq n \end{cases}$$

$$s_i(\sigma,\ \Phi) = \begin{cases} (s_i\,\sigma,\ \Phi) & 0 \leq i \leq p \\ (\sigma,\ s_{i-p-1}\,\Phi) & p < i \leq n, \end{cases}$$

(where $p = \dim \sigma$*) whenever this makes sense, and* $d_i(\sigma,\ \Phi) = *$ *otherwise.*

Note that for X a simplicial set there is a canonical homeomorphism between $|EX|$ and the topological supension $S\,|X|$ ([6], Proposition 2.3) and for Y a Kan simplicial space for which LY is defined, $|LY|$ is homotopy equivalent to the topological loop space $\Omega\,|Y|$.

3.3. PROPOSITION. *There is a continuous natural isomorphism* $\Theta : (EX,\ Y)_A \to (X,\ LY)_A$ *where* X *is a simplicial set and* Y *a Kan simplicial space. This isomorphism passes to homotopy classes to give a natural isomorphism* $[X,\ LY]_B \cong [EX,\ Y]_B$.

Indeed, Θ is given by $\Theta f(x) = f(x,\ \Phi_0)$ where $f : EX \to Y$ and $x \in X_{n-1}$ and $f(x,\ \Phi_0)$ is regarded as an element of $(LY)_{n-1}$.

3.4. PROPOSITION. *Let* Y *be a Kan simplicial compact Hausdorff space, with* $*$ *the only vertex in its path component. Then* $[\ ,\ |LY|\,]_C$ *is a compact Hausdorff group valued functor on the homotopy category of CW complexes.*

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* If Y is not connected but $*$ is still the only vertex in its path component, then by LY we mean the functor L applied to that component only.

Proof. By Lemma 2.7 and Fact 2.16 we need only show that $[X, LY]_B$ is a topological group when X is a finite simplicial set. Look instead at $[EX, Y]$ which is naturally homeomorphic to $[X, LY]_B$ under a homeomorphism preserving the group structure. Take simplicial approximations μ and ε to the obvious maps $|EX| \to |EX| \vee |EX|$ and $|EX| \xrightarrow{-1} |EX|$ say $\mu : (EX)' \to EX \vee EX$ and $\varepsilon : (EX)'' \to EX$. There are simplicial maps $\varphi : (EX)' \to EX$ and $\psi : (EX)'' \to EX$ each of which is homotopic to a homeomorphism. Therefore φ^* and ψ^* are homeomorphisms. The multiplication and inverse on $[EX, Y]$ are then the compositions

$$[EX \vee EX, Y] \cong [EX, Y] \times [EX, Y] \xrightarrow{\mu^*} [(EX)', Y] \overset{(\varphi^*)^{-1}}{\underset{\cong}{\longrightarrow}} [EX, Y]$$

and

$$[EX, Y] \xrightarrow{\varepsilon^*} [(EX)'', Y] \overset{(\psi^*)^{-1}}{\underset{\cong}{\longrightarrow}} [EX, Y]$$

which being compositions of induced maps are continuous.

3.5. DEFINITION. *A Kan simplicial compact Hausdorff L-prespectrum consists of a sequence* $\{K_n, \varkappa_n\}_{n \geq 0}$ *of Kan simplicial compact Hausdorff spaces and continuous maps* $\varkappa_n : K_n \to LK_{n+1}$ *so that* $\varkappa_n$ *are isomorphisms of simplicial spaces.*

3.6. PROPOSITION. *Let* $\{K_n, \varkappa_n\}_{n \geq 0}$ *be a Kan simplicial compact Hausdorff L-prespectrum. Then* $\{[\ \ , |K_n|]_c\ \ , (\varkappa_n)_*\}_{n \geq 0}$ *is a generalized cohomology theory on the homotopy category of CW complexes with values in the category of compact Hausdorff abelian groups.*

Proof. The functor $[\ \ , |K_n|]_c$ is compact Hausdorff group valued by Proposition 3.4. Now the homotopy, exactness and suspension axioms follow from the fact that they hold algebraically, all the arrows are defined via induced maps and the maps $(\varkappa_n)_*$, and that $[X, |K_n|]_c$ are compact Hausdorff spaces.

The last proposition leads one to make the following definition.

3.7. DEFINITION. *A compact Hausdorff generalized cohomology theory* $\{\mathscr{K}^*, \varkappa^*\}$ *on the homotopy category of CW complexes is said to be representable if there is a Kan simplicial compact Hausdorff L-prespectrum* $\{K_n, \varkappa_n\}$ *such that* K_n *represents* $\mathscr{K}^n$ *in the sense of Definition 2.13, and the suspension transformations are induced by the maps* $\varkappa_n$.

3.8. EXAMPLE. Let G be a compact Hausdorff abelian group. Then the ordinary cohomology HG^* with coefficients in G has a natural compact Hausdorff topology [5]. As remarked in Section 2, HG^n is represented by $EM(G, n)$.

Claim. One can define maps $\varepsilon_n : EM(G, n) \to LEM(G, n+1)$ so that $\{EM(G, n), \varepsilon_n\}_{n \geq 0}$ form an L-prespectrum.

Indeed, the appropriate maps on the chain level $C_n(\Delta[q]) \to$ $\to C_{n+1}(\Delta[q+1])$ induce maps $\eta_n : LEM(G, n+1) \to EM(G, n)$ which are isomorphisms of simplicial compact Hausdorff spaces for they are

i) maps of simplicial sets

ii) continuous and

iii) both $EM(G, n)$ and $LEM(G, n+1)$ are minimal.

Now take $\varepsilon_n = \eta_n^{-1}$.

4. Representability of compact Hausdorff cohomology theories

In this section we prove the following

4.1. THEOREM. *Given a generalised cohomology theory* $\mathcal{Y}^*$ *on the homotopy category of CW complexes with values in the category of compact Hausdorff abelian groups and continuous homomorphisms, there is a sequence of Kan simplicial compact Hausdorff spaces* $\{Y_n\}$ $n \geq 0$ *such that* $\mathcal{Y}^*$ *is represented in the sense of Definition 2.13 by* Y_n.

We shall use some facts announced in [3] that we list below.

Let Ch be the Pontrjagin duality functor from the category of locally compact Hausdorff abelian groups to itself, defined by $Ch\,G = \mathrm{Hom}(G, R/Z)$, where R/Z is the additive group of reals modulo the integers with the usual topology. The functor Ch has the following properties.

(i) Ch is exact

(ii) $Ch \varinjlim G_\alpha = \varprojlim Ch\, G_\alpha$, where G_α are discrete groups

(iii) Ch sends compact groups to discrete ones and vice versa

(iv) $Ch \circ Ch \cong Id$

For any topological spectrum $\mathcal{Y}$ let $\mathcal{Y}^*$ and $\mathcal{Y}_*$ denote the cohomology and homology theories associated with it. A script letter will be used for a topological spectrum and an italic one for an L-prespectrum.

Given a spectrum $\mathcal{Y}$ it is easy to see using (i), (ii) and (iii) that $Ch \circ \mathcal{Y}_*$ is an additive cohomology theory with values in the category of compact Hausdorff abelian groups. Let $\mathcal{Y}$ denote the spectrum corresponding to $U \circ Ch \circ \mathcal{Y}_*$, where U is the functor that forgets the topology.

Conversely, let F^* be a compact Hausdorff group valued cohomology theory. Then for any CW complex X, $F(X) = \varprojlim F^*(X_\alpha)$ where X_α runs over the finite subcomplexes of X (to see this, observe that the right hand side, considered as a functor of X satisfies the wedge and Mayer-Vietoris axioms and is therefore representable; applied to the spheres it agrees with F^*, so by a theorem of J. H. C. Whitehead it is naturally equivalent to F^*). Therefore $Ch \circ F^*$ is an additive homology theory corresponding to a spectrum $\mathcal{G}$. Using (*iv*) it is clear that F^* is $Ch \circ \mathcal{G}_*$.

Let $\mathcal{S}$ denote the sphere spectrum.

Suppose we find a Kan simplicial compact Hausdorff L-prespectrum $\{S_n^c, \sigma_n^c\}_{n \geq 0}$ representing in the sense of Definition 3.7 $Ch \circ \mathcal{S}_*$. We claim that one can then construct a sequence $\{Y_n^c\}_{n \geq 0}$ of Kan simplicial compact Hausdorff spaces representing each term of a given compact Hausdorff cohomology theory $\mathcal{G}^{c*}$.

Indeed, given a compact Hausdorff cohomology theory $\mathcal{G}^{c*}$, take a representing Kan simplicial set prespectrum for $\mathcal{G}_* = Ch \circ \mathcal{G}^{c*}$. This can be done by taking a topological Ω-spectrum corresponding to $\mathcal{G}_*$ and then applying the functor »singular complex« (for details see [6]). We get a Kan simplicial set prespectrum $\{Y_n, \eta_n\}_{n \geq 0}$.

If A is a simplicial set and B is a pointed simplicial set, let their semi-smash product $A \cdot B$ be defined as in ([6] Definition 6.4). There is a map $j: A \cdot EB \to E(A \cdot B)$ which is natural and an isomorphism ([6], Proposition 6.6).

Further, if both A and B are pointed simplicial sets, let $A \wedge B$ be the simplicial set obtained from $A \times B$ by identifying the simplices of the form $(a, *)$ and $(*, b)$ with the appropriate degeneracy of the base point. Let $A \cdot B$ denote the simplicial set obtained by performing Kan's identifications ([6], Definition 6.4) on $A \wedge B$. This does not change the homotopy type of $A \wedge B$ since (in the notation of [6]) t_j are inclusions and y is a homotopy equivalence.

Let $(S_{k+r}^c)^{Y_r}$ be the simplicial space with

$((S_{k+r}^c)^{Y_r})_q$ = space of simplicial maps $\Delta[q] \cdot Y_r \to S_{k+r}^c$

with usual face and degeneracy operators induced by $\Delta[q-1] \to \Delta[q]$ and $\Delta[q+1] \to \Delta[q]$ ($\Delta[q]$ is of course unpointed). A suitable modification of the proof of ([7], Theorem 6.9) shows that $(S_{k+r}^c)^{Y_r}$ is Kan.

Now for each r define a map $(S_{k+r}^c)^{Y_r} \xrightarrow{\psi} (S_{k+r-1}^c)^{Y_{r-1}}$ by $\psi(f) = (\sigma_{k+r-1}^c)^{-1} \circ$ adjoint of $[f \circ (1 \cdot \text{adjoint of } \eta_{r-1}) \circ j^{-1}]$ where $f \in ((S_{k+r}^c)^{Y_r})_q$, $\sigma^{-1}: L S_{k+r}^c \to S_{k+r-1}^c$. Take Y_k^c to be the inverse limit of the inverse system $\{(S_{k+r}^c)^{Y_r}, \psi\}_{r \geq 0}$.

If A is a pointed simplicial set, it follows from the functoriality of $*\cdot *$ and the definition of a simplicial set that there is an isomorphism $(A\cdot Y_r, S^c_{k+r}) \cong (A, (S^c_{k+r})^{Y_r})$. Therefore

$$\begin{aligned}
&[A, \varprojlim_r (S^c_{k+r})^{Y_r}] \cong \varprojlim_r [A, (S^c_{k+r})^{Y_r}] \\
&\cong \varprojlim_r [A \wedge Y_r, S^c_{k+r}] \cong \varprojlim_r Ch\circ \pi_{k+r}(A \wedge Y^c_r) \\
&\cong Ch\circ \varinjlim_r \pi_{k+r}(A \wedge Y^c_r) \\
&\cong Ch\circ Ch\circ (\mathscr{Y}^c)^k\,(|A|) \\
&\cong (\mathscr{Y}^c)^k\,(|A|)
\end{aligned}$$

Before passing to the construction of the Kan simplicial compact Hausdorff L-prespectrum $\{S^c_n,\ \sigma^c_n\}_{n\geq 0}$ representing $Ch\circ \mathscr{S}_*$, we need one more result from [3].

4.2. *Fact*. If $\mathscr{Y}$ and $\mathscr{Y}^c$ are topological Ω-spectra corresponding to $\mathscr{Y}_*$ and $\mathscr{Y}^*Ch\circ \mathscr{Y}_*$, then there is a natural algebraic isomorphism $U\circ Ch\,\pi_i(\mathscr{Y}) \cong \pi_{-i}(\mathscr{Y}^c)$.

The proof is easy:

$$\pi_{-i}(\mathscr{Y}^c) \cong \mathscr{Y}^{c^i}(S^0) \cong U\circ Ch\ \mathscr{Y}_i(S^0) \cong U\circ Ch\,\pi_i(\mathscr{Y}).$$

4.3. *Construction of* S^c. It is a well-known fact that the homotopy groups of the sphere spectrum are finite for $i\geq 1$ and that $\pi_0(\mathscr{S}) \cong Z$.

Now the Pontrjagin dual of a finite group is a finite group and $Ch\,Z \cong R/Z$.

By fact 4.2. $\mathscr{S}^c$ has non-trivial homotopy groups only in non-positive dimensions and they are all finite but for

$$\pi_0(\mathscr{S}^c) \cong Ch\,Z \cong R/Z.$$

Consider the Postnikov fibre $\bar{\mathscr{S}}_*$ of the homology theory $\mathscr{S}_*$ got by killing the only non-finite homotopy group of $\mathscr{S}$.

We shall first show that there is a Kan simplicial compact Hausdorff L-prespectrum representing in the sense of Definition 3.7 the cohomology theory $Ch\circ \bar{\mathscr{S}}_*$.

In order to do this, take a topological Ω-spectrum representing $U\circ Ch\circ \bar{\mathscr{S}}_*$. Apply the singular complex functor to that Ω-spectrum to get a Kan simplicial set prespectrum [6]. Choose a minimal one in it, i. e. for each n choose a minimal Kan simplicial set that is a strong deformation retract of the n-th term of the above Kan simplicial prespectrum, and take the obvious structure maps.

4.4. *Claim.* The resulting object $\{\tilde{S}^c_n, \tilde{\sigma}^c_n\}_{n\geq 0}$ is a Kan simplicial finite set L-prespectrum.

Proof. The simplicial set $\tilde{S}^c_n$ has finite homotopy groups and is minimal. Therefore $\tilde{S}^c_n$ is a simplicial finite set: use [8] Moore's way of constructing it and the fact that $EM(F, k)$ is a simplicial finite set if F is finite.

It remains to show that the structure maps $\tilde{\sigma}^c_n : \tilde{S}^c_n \to L\tilde{S}^c_{n+1}$ are isomorphisms.

But the Kan simplicial set $L\tilde{S}^c_{n+1}$ is minimal since $\tilde{S}^c_{n+1}$ was so. Now the structure map $\tilde{\sigma}^c_n : \tilde{S}^c_n \to L\tilde{S}^c_{n+1}$ is a homotopy equivalence since we started off with a topological Ω-spectrum. Since both $\tilde{S}^c_n$ and $L\tilde{S}^c_{n+1}$ are minimal Kan complexes, $\tilde{\sigma}^c_n$ is an isomorphism ([7], Proposition 9.7).

Since $\tilde{S}^c_n$ is a Kan simplicial finite set, we can consider it as a Kan simplicial compact Hausdorff space, with the discrete topology on each $(\tilde{S}^c_n)_k$. We thus obtain a Kan simplicial compact Hausdorff L-prespectrum $\{\tilde{S}^c_n, \tilde{\sigma}^c_n\}_{n\geq 0}$.

4.5. LEMMA. *$\{\tilde{S}^c_n, \tilde{\sigma}^c_n\}_{n\geq 0}$ represents $Ch\cdot\tilde{\mathscr{S}}_*$ in the sense of Definition 3.7.*

Proof. By Fact 2.16 it is enough to prove the assertion for finite CW complexes X. In that case $[X, |\tilde{S}^c_n|]_c$ is finite and compact Hausdorff i. e. discrete as indeed it should be.

We now want to add the missing homotopy group to $\tilde{S}^c$.

Let $\mathscr{A}_n$ denote the compact Hausdorff space of maps

$$\tilde{S}^c_n \to EM(R/Z, n+1)$$

representing the last k-invariant of the n-th term of the Ω-spectrum corresponding to $Ch\cdot\mathscr{S}_*$.

Since $\tilde{S}^c_n$ is discrete, any map $\tilde{S}^c_n \to EM(R/Z, n+1)$ is continuous.

There is a continuous map $\mathscr{A}_n \to \mathscr{A}_{n-1}$ given by $f \mapsto Lf$. The inverse limit $\varprojlim_n \mathscr{A}_n$ is non-empty by Fact 2.15. Take an element $(f_n)_{n\geq 0}$ in it. Take the pull back S^c_n in the category of simplicial spaces of the path fibration

$$P(EM(R/Z, n+1)) \to EM(R/Z, n+1)$$

by the map f_n.

4.6. *Claim.* The resulting object S^c_n is a Kan simplicial compact Hausdorff space.

Proof. For the extension property see [7]. Compactness and Hausdorffness follow from S^c_n being a subspace, determined by an equation, of a product of two compact Hausdorff spaces.

4.7. *Claim.* For each n, there is an isomorphism $\sigma^c_n : S^c_n \to L S^c_{n+1}$ of simplicial spaces.

Proof. The space $L S^c_n$ is the pull-back of $L\,PEM(R/Z,\ n+1) \xrightarrow{L_{PEM}} LEM(R/Z,\ n+1)$ by $L f_n$, and to show that $L S^c_n$, $n \geq 1$ is isomorphic to S^c_{n-1} (which is the pull-back of $P\,LEM(R/Z,\ n+1) \xrightarrow{P_{LEM}} LEM(R/Z,\ n+1)$ by $L f_n = f_{n-1}$) it is enough to give an isomorphism $L\,PEM(R/Z,\ n+1) \xrightarrow{\cong} P\,LEM(R/Z,\ n+1)$ over $LEM(R/Z,\ n+1)$. Recall that $(EM(R/Z,\ n+1))_q = Z^{n+1}(\Delta[q],\ R/Z)$. Define a map $g_q : \Delta[q] \to \Delta[q]$ by $(0,\ 1, \ldots, q-1,\ q) \mapsto (0,\ 1, \ldots, q,\ q-1)$. This induces an isomorphism $g^* : EM(R/Z,\ n+1) \to EM(R/Z,\ n+1)$ which is not simplicial but commutes with all but the last two face and degeneracy operators, and such that $d_{q-1} = d_q \circ g^*_q$, $d_q = d_{q-1} \circ g^*_q$. Therefore g^* defines a simplicial isomorphism

$L\,PEM(R/Z,\ n+1) \xrightarrow{\cong} P\,LEM(R/Z,\ n+1)$ over $LEM(R/Z,\ n+1)$.

Take these maps σ^c_n as structure maps of the L-prespectrum we are constructing.

To complete the proof of the theorem it remains to show that for each CW complex X, $Ch \circ \mathscr{S}_*(X)$ and $[X,\ |S^c_*|]_r$ are isomorphic as topological groups in a natural way.

4.8. LEMMA. *There is a unique homology theory h_* s. t.*

$$\ldots \to h_* \to \mathscr{H}Z_* \to \bar{\mathscr{S}}_{*-1} \to \ldots \text{ is exact.}$$

Note. Here $\mathscr{H}Z_*$ denotes ordinary homology with coefficients Z (compare Example 2.14). The script letter is used conforming to the notation of this section.

Proof. Since the homotopy groups of $\bar{\mathscr{S}}$ are finite, there is a natural isomorphism

$$\lim_{\substack{\leftarrow \\ X_f}} [X_f,\ \bar{\mathscr{S}}_*] \cong [X,\ \bar{\mathscr{S}}_*] \qquad (*)$$

for each CW complex X, where X_f runs over the finite subcomplexes of X [9].

By using Spanier Whitehead duality as is done in Section 8 of [10], one sees that the cohomology theories h^*, $\bar{\mathscr{S}}^*$ and $\mathscr{H}Z^*$ corresponding to the representing spectra h, $\mathscr{S}$, $\mathscr{H}Z$

of h_*, $\tilde{\mathscr{S}}_*$, $\mathscr{H} Z_*$ but defined on finite CW complexes only, are linked in the following exact sequence

$$\ldots \to h^* \to \mathscr{H} Z^* \to \tilde{\mathscr{S}}^{*+1} \to \ldots$$

It follows from the isomorphism (*) that the natural transformation $\mathscr{H} Z^* \to \tilde{\mathscr{S}}^{*+1}$ of functors defined on the homotopy category of finite complexes is induced by a map of spectra $\mathscr{H} Z \to \tilde{\mathscr{S}}$.

Take the fibre $\mathscr{F}$ of the map $\mathscr{H} Z \to \tilde{\mathscr{S}}$. Again, by the isomorphism (*) the composite transformation

$$h^* \to \mathscr{H} Z^* \to \tilde{\mathscr{S}}^{*+1}$$

is induced by a map of spectra $h \to \tilde{\mathscr{S}}$ and that map is trivial. Therefore, there is a map $h \to \mathscr{F}$ which is easily shown using the five lemma and J. H. C. Whitehead's theorem to be a homotopy equivalence.

Given a compact Hausdorff cohomology theory G^* there is an exact sequence ([4], p. 27)

$$\ldots \to G[m]^{q-1} \to H^{q+m+1}(\quad ;\ G^{-m-1}(S^0)) \to G[m+1]^q \to$$
$$\to G[m]^q \to$$

where $G[m]^*$ are the Postnikov factors of G^* defined by

$$G[m]^q(x) = \operatorname{Im}(G^q(x^{m+q+1}) \to G^q(x^{m+q}))$$

(here X^l denotes the l-skeleton of X) and all the arrows are continuous natural transformations since they are defined via induced maps and the suspension transformations [4].

Applying this to $Ch \circ \mathscr{S}_*$ and taking $m = -1$ in the above, we get the exact sequence (with notation as in 2.14)

$$\ldots \to Ch \circ \tilde{\mathscr{S}}_{q-1} \to \mathscr{H} R/Z^q \to Ch \circ \mathscr{S}_q \to Ch \circ \tilde{\mathscr{S}}_q \to \ldots$$

The sequence $EM(\quad, R/Z) \to S^c \to \tilde{S}^c$ of Kan simplicial compact Hausdorff L-prespectra induces an exact sequence

$$\ldots \to [\quad, |S_q^c|] \to [\quad, |\tilde{S}_q^c|] \to [\quad, EM(q+1, R/Z)] \to \ldots$$

By the construction of S^c there is a commutative diagram

$$\begin{array}{ccccccc} \ldots \to Ch \circ \mathscr{S}_q & \to & Ch \circ \tilde{\mathscr{S}}_q & \to & \mathscr{H} R/Z^{q+1} \to \ldots \\ & & i \downarrow & & \downarrow k \\ \ldots \to [\quad, |S_q^c|] & \to & [\quad, |\tilde{S}_q^c|] & \to & [\quad, |EM(q+1; R/Z)|] \to \ldots \end{array}$$

where i is a natural equivalence by Lemma 4.5 and k is a natural equivalence by Example 2.14.

Now apply Ch to the diagram above and use Lemma 4.8 to complete the proof of the theorem.

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ТОПОЛОШКИ ОБЈЕКТИ ВО ТЕОРИЈА НА ХОМОТОПИЈА

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Резиме

Под тополошки објект во една категорија подразбираме објект X во таа категорија заедно со топологија на множествата $\mathrm{Mor}(Y, X)$ за секое $Y \in \mathrm{Ob}\, C$ така што $\mathrm{Mor}(\ \ , X)$ на тој начин дефинира функтор од C во категоријата на тополошки простори и непрекинати пресликувања.

Нека C е категоријата од CW комплекси и хомотопски класи на пресликувања. Нека $X, Y \in \mathrm{Ob}\, C$ и нека со $[Y, X]$ го означиме $\mathrm{Mor}_C(Y, X)$. Предложена е дефиниција за репрезентабилноста на функтор $[\ \ , X] : C \to$ категорија на компактни

Хаусдорфови простори и непрекинати пресликувања, со тоа што X е снабдено со додатна структура од која може да се добие топологијата на $[Y, X]$ за секое Y.

Покажано е дека Канов симплицијален компактен Хаусдорфов L-преспектрум (симплицијалниот објект кој одговара на тополошки Ω-спектрум но со додатната структура спомната погоре) дефинира генерализирана теорија на кохомологија со значења во категоријата на компактни Хаусдорфови абелови групи и непрекинати хомоморфизми. На крајот репрезентабилноста на истите е докажана.