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P-LOCALIZATION IN pro-Ab

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Abstract. The completeness of pro-A for any abelian category A and co-completeness of pro-A for any co-complete abelian category A are proved. For a given set P of primes and a given co-complete abelian category A, a P-localization functor in A is defined. Combining these results, one obtains the existence of a P-localization functor in pro-Ab, where Ab is the category of abelian groups. This functor restricts on Ab to the usual one, but differs from the »level-wise« application of the P-localization functor in Ab.

1. Introduction

The theory of localization at a set P of primes has been introduced in the homotopy category H of CW complexes and much work has been done on its applications (see for example [3]). On the other hand, we now witness a tendency to find shape theoretic analogues of techniques used in classical homotopy theory and to prove such analogues of classical theorems. Usually, however, the results are easier to obtain in pro-H, as defined in [1]. But to be able to introduce P-localization in pro-H, one has first got to consider P-localization in pro-Ab, where Ab is the category of abelian groups. That is the aim of this note.

In Section 2 we collect some well-known facts about pro-categories. In Section 3 we define the P-localization in an abelian category and prove its existence in any co-complete abelian category. The completeness and co-completeness of pro-Ab is proved in Section 4. I wish to express my gratitude to J. Vrabec for a helpful discussion.

2. Pro-categories

2.1. DEFINITIONS. A category I is said to be *cofiltering* if it satisfies the following two conditions:

(i) (cone condition) For every pair of objects $i, i' \in I$, there is a diagram in I of the form

$$\begin{array}{ccc} i & & \\ & \nwarrow & \\ & & i'' \\ & \swarrow & \\ i' & & \end{array}$$

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(*ii*) (essential uniqueness of morphisms) If $i' \rightrightarrows i$ is a diagram in I, there is a morphism $i'' \to i'$ in I such that the two compositions $i'' \to i$ are equal.

Let C be a category. A pro-*object*

$$\mathbf{X} = \{X_i\}_{i \in I}$$

in C is an I-diagram in C (i.e. a functor from I to C), where I is a small cofiltering category.

Define a category pro-C as follows. The objects of pro-C are the pro-objects in C. If $\mathbf{X} = \{X_i\}_{i \in I}$ and $\mathbf{Y} = \{Y_j\}_{j \in J}$ are two objects of pro-C, the set of morphisms from $\mathbf{X}$ to $\mathbf{Y}$ in pro-C is given by

$$[\mathbf{X}, \mathbf{Y}]_{\text{pro-}C} = \lim_J \operatorname{colim}_I \, [X_i, Y_j]_C.$$

A morphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ in pro-C is represented by a map (of sets) $\varphi\colon J \to I$ and for each $j \in J$ a morphism

$$f_j\colon X_{\varphi(j)} \to Y_j$$

in C, such that if $j' \to j$ is in J, then there is in I a diagram

$$\begin{array}{ccc} & & \varphi(j) \\ & \nearrow & \\ i & & \\ & \searrow & \\ & & \varphi(j') \end{array}$$

with

$$\begin{array}{ccccc} & & X_i & & \\ & \swarrow & & \searrow & \\ X_{\varphi(j)} & & & & X_{\varphi(j')} \\ {\scriptstyle f_j}\downarrow & & & & \downarrow{\scriptstyle f_{j'}} \\ Y_j & & \longleftarrow & & Y_{j'} \end{array}$$

commutative. Two pairs (φ, f_j) and (φ', f'_j) represent the same morphism in pro-C if for each $j \in J$ there is a diagram

$$\begin{array}{ccc} & & \varphi(j) \\ & \nearrow & \\ i & & \\ & \searrow & \\ & & \varphi'(j) \end{array}$$

in I, with

$$\begin{array}{ccc} & X_i & \\ \nearrow & & \nwarrow \\ X_{\varphi(j)} & & X_{\varphi'(j)} \\ {\scriptstyle f_j}\searrow & & \swarrow{\scriptstyle f'_j} \\ & Y_j & \end{array}$$

commutative.

We can consider C as a full subcategory of pro-C: an object of C is a pro-object in C with index category the one-point category.

A functor $F: C \to Set$ from C to the category of sets is said to be pro-*representable* if there is a pro-object $\mathbf{X}$ in C such that F is naturally isomorphic to $[\mathbf{X}, -]_{\text{pro-}C}$.

A functor $F: C \to Set$ is called *proper* if in C there is a set D of objects with the following property: if Y is any object of C and if $y \in F(Y)$, then for some object Z in D there is a $z \in F(Z)$ and a morphism $f: Z \to Y$ with $y = F(f)(z)$.

2.2. PROPOSITION. *Let C be finitely complete. Then a functor $F: C \to Set$ is pro-representable if and only if F is proper and preserves finite limits.* ([4], 10.7.6. and [1], Appendix, Prop. 2.7.)

2.3. PROPOSITION. ([1], Appendix, Prop. 4.4) *For any category C, pro-C is closed under small cofiltering limits.*

2.4. PROPOSITION. ([1], Appendix, Prop. 4.5) *If A is an abelian category, then so is* pro-A.

3. *P*-localization in an abelian category

3.1. DEFINITION. *Let P be a set of primes and P' be the set of primes not in P. Let A be an abelian category. An object X of A is said to be P-local if for each $p \in P'$, the morphism $p_X : X \to X$* (p times the dentity) *is an isomorphism.*

Denote by *P-A* the full subcategory of A generated by P-local objects. Thus, *P-Ab* is the category of P-local abelian groups.

Let Z_P be the subring of the rationals, Q, consisiting of those rationals a/b where b is a product of powers of primes in P'.

The category *P-Ab* is just the category of Z_P-modules ([2], Prop. 2.2). Indeed, a Z_P-module is a P-local abelian group, any P-local abelian group admits a unique structure of a Z_P-module and any homomorphism between P-local abelian groups is Z_P-linear.

Recall that if A is an abelian category, X an object of A, R a commutative ring with unit and $f : R \to [X, X]_A$ a unit preserving ring homomorphism, we say that f gives X the structure of an *R-object* in A.

Proposition 2.2. from [2] easily generalizes to any abelian category A:

3.2. PROPOSITION. *For any abelian category A, P-A is the category of Z_P-objects in A, i.e. any Z_P-object in A is P-local, any P-local object in A admits a unique structure of a Z_P-object and any morphism between P-local objects preserves their Z_P-structure.*

3.3. DEFINITION. *Let A be an abelian category and P-A as above. A left adjoint functor L_P to the inclusion functor $U: P\text{-}A \to A$ is called a P-localization in A.*

L_P is obviously determined up to a natural isomorphism, and it immediately follows that L_P preserves colimits.

It is well-known [4] that the unit of the adjunction is a natural transformation $e : I \to UL_P$, where I is the identity functor. For an object X of A, we say that the morphism $e_X: X \to UL_P(X) = L_P(X)$ *P-localizes X.*

This morphism e_X has the following universal property: *given any P-local object Y in A and a morphism $f : X \to Y$, there is a unique morphism g_a: $L_P(X) \to Y$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{e_X} & L_P(X) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ Y & & \end{array}$$

commutes.

If $A = Ab$ one can prove the existence of L_P for any set P of primes by using the special adjoint functor theorem ([4], Prop. 16.4.7.) since P-Ab is easily seen to be complete and well-powered and to have a co-generator (namely Z_P/Z, where Z denotes the integers) and U preserves limits. Or, alternatively, a direct computation as in [2], Prop. 2.1, shows that $e_X: X \to Z_P \otimes X$ given by $x \mapsto 1 \otimes x$ P-localizes X.

To prove the existence of P-localization in pro-Ab, the first argument does not seem to work for I do not know whether (and doubt that) pro-Ab is well-powered and possesses a co-generating set. But one can follow the line of the second argument and that is what we shall do now.

If A is a co-complete abelian category and R is an abelian group, there is a functor

$$R \otimes -: A \to A$$

with a natural isomorphism

$$\Theta = \Theta_{X,Y} : [R \otimes X, Y]_A \to [R, [X, Y]_A]_{Ab} \tag{3.3}$$

(see [4], 17.7.6).

Taking $R = Z_P$ and $Y = Z_P \otimes X$ in (3.3) let $h_X : X \to Z_P \otimes X$ denote the image of $1 \in Z_P$ under $\Theta(1_{Z_P \otimes X})$. We shall prove that h_X P-localizes X.

Step 1. $Z_P \otimes X$ is *P-local.*

Proof. For any $p \in P'$, $p_{Z_P}: Z_P \to Z_P$ is an isomorphism, so

$$p^*_{Z_P}: [Z_P, [X, Y]] \to [Z_P, [X, Y]]$$

is an isomorphism for each Y. The diagram

$$\begin{array}{ccc} [Z_P \otimes X, Y] & \xrightarrow[\cong]{\Theta} & [Z_P, [X, Y]] \\ {\scriptstyle p^*_{Z_P}} \downarrow & & \downarrow {\scriptstyle p^*_{Z_P}} \\ [Z_P \otimes X, Y] & \xrightarrow[\cong]{\Theta} & [Z_P, [X, Y]] \end{array}$$

commutes, so $p_{Z_P \otimes X}$ is an isomorphism.

Step 2. *If X is P-local, h_X is an isomorphism.*

Proof. If X is P-local, then there is a unique unit preserving ring homomorphism $\varphi : Z_P \to [X, X]$. Denote $\Theta^{-1}(\varphi) : Z_P \otimes X \to X$ by k_X. We shall prove that h_X and k_X are mutually inverse.

To show that $h_X k_X = 1_{Z_P \otimes X}$, chase φ in the diagram

$$\begin{array}{ccc} [Z_P, [X, X]] & \xrightarrow{\Theta^{-1}} & [Z_P \otimes X, X] \\ {\scriptstyle (h_X)_*} \downarrow & & \downarrow {\scriptstyle (h_X)_*} \\ [Z_P, [X, Z_P \otimes X]] & \xrightarrow{\Theta^{-1}} & [Z_P \otimes X. Z_P \otimes X]. \end{array}$$

To show that $k_X h_X = 1_X$, chase $1_{Z_P \otimes X}$ in the diagram

$$\begin{array}{ccc} [Z_P \otimes X, Z_P \otimes X] & \xrightarrow{\Theta} & [Z_P, [X, Z_P \otimes X]] \\ {\scriptstyle (k_X)_*} \downarrow & & \downarrow {\scriptstyle (k_X)_*} \\ [Z_P \otimes X, X] & \xrightarrow{\Theta} & [Z_P, [X, X]]. \end{array}$$

Step 3. (*Universality of* h_X) *Let* $f : X \to Y$ *be a morphism in* A *with* Y *P-local. Then* $g : Z_P \otimes X \to Y$ *given by*

$$g = (h_Y)^{-1} (1 \otimes f)$$

is the unique morphism that satisfies $f = gh_X$.

We have thus proved the following

3.4. THEOREM. *Let* A *be a co-complete abelian category. Then for any set* P *of primes there is a P-localization functor in* A.

In Section 4 we shall prove the co-completeness of pro-A for any co-complete abelian category A. So, in particular, avery pro-abelian-group admits a P-localization.

3.5. *Remark.* The P-localization of an abelian group X in Ab coincides with the P-localization of X (considered as a pro-abelian-group) in pro-Ab. But it is not in general true that the P-localization of $\{X_i\}_{i\in I}$ is $\{Z_P \otimes X_i\}_{i\in I}$. For example, consider the pro-abelian-group

$$\mathbf{X} = Z \overset{2}{\leftarrow} Z \overset{2}{\leftarrow} Z \overset{2}{\leftarrow} \dots$$

and let P be the set of odd primes. Then $\mathbf{X}$ is P-local and not isomorphic to

$$Z_P \overset{2}{\leftarrow} Z_P \overset{2}{\leftarrow} Z_P \overset{2}{\leftarrow} \dots$$

4. Completeness and co-completeness of pro-Ab

4.1. PROPOSITION. *Let* A *be an abelian category. Then* pro-A *is complete.*

Proof. Since pro-A is abelian (Prop. 2.4), it has finite products. Moreover, it is closed under limits over small cofiltering categories (Prop. 2.3). Now, if a category C is closed under finite products and limits over small cofiltering categories, it has all small products. For, if $\{X^j\}_{j\in J}$ is a set of objects of C, let J_f denote the codirected set of finite subsets of J ordered by inclusion. Then the limit of the system

$$\left\{ \prod_{j\in s} X^j \right\}_{s\in J_f}$$

with projections as bonding morphisms is the product of $\{X^j\}_{j\in J}$ in C.

The completeness of pro-A now follows from the facts that it has all small products and, being abelian, has equalizers.

4.2. PROPOSITION. *Let A be a co-complete abelian category. Then* pro-A *is co-complete.*

Proof. Since pro-A is abelian, it has coequalizers. It is therefore sufficient to prove that pro-A has all small coproducts, i.e. that for a given set $\{\mathbf{X}^j\}_{j\in J}$ (where $\mathbf{X}^j = \{X_i^j\}_{i\in I_j}$) of objects of pro-$A$, the functor

$$F = \prod_{j\in J} [\mathbf{X}^j, -]\colon\ A \to Set$$

preserves finite limits and is proper. The first claim follows from the fact that in the category of sets, finite limits commute with filtering colimits.

To prove that F is proper, consider the set

$$D = \{Z^t\colon\ t \in \prod_j I_j\}$$

of objects in pro-A where for $t = (i_j)_{j\in J}$, Z^t is the coproduct of the $X_{i_j}^j\ (j \in J)$.

Let Y be an arbitrary object of A and let $y \in F(Y)$, $y = (\mathbf{y}_j)_{j\in J}$. For each $j \in J$, let the morphism $\mathbf{y}_j$ in pro-A be represented by morphisms

$$y_j\colon\ X_{i_j}^j(y) \to Y$$

in A. Put $t = (i_j(y))_{j\in J}$. Let $z \in F(Z^t)$ be determined by the inclusions

$$X_{i_j(y)}^j \to Z^t$$

and $f\colon\ Z^t \to Y$ be determined by the morphisms y_j. Then $y = F(f)(z)$ which shows that F is proper.

4.3. *Remark.* The previous proposition shows that the coproduct of the family

$$\{\{X_i^j\}_{i\in I_j}\}_{j\in J}$$

is, in the above notation, given by $\{Z^t\}_{t\in\prod I_j}$. (Specially, if $\{X^j\}_{j\in J}$ are objects of C, then their coproduct in C coincides with their coproduct in pro-C). One could prove this directly using the following

4.4. LEMMA. *Let J be a set and for each $j \in J$ let I_j be a diagram scheme. Let*

$$A^j\colon\ I_j \to Set$$

be I_j-diagrams in Set, sending i_j to $A_{i_j}^j$. Form the diagram

$$\prod A^j\colon\ \prod I_j \to Set$$

given by

$$(i_j) \in \prod_j I_j \mapsto \prod_j A^j_{i_j}.$$

Then there is a natural bijection

$$\prod_j \operatorname{colim}_{I_j} A^j_{i_j} \to \operatorname{colim} \prod_j A^j_{i_j}$$

where the right hand colimit is over $\prod_j I_j$. ■

In the special case where the diagram schemes I_j are discrete, this lemma just states the distributivity of the product over the disjoint union.

4.5. *Remark.* It is not true in general that the product of $\{X^j\}_{j\in J}$, where X^j are objects in C, in pro-C coincides with their product in C (if both exist).

REFERENCES:

[1] *M. Artin* and *B. Mazur*, Etale homotopy, LNM 100, Springer Berlin 1969.

[2] *P. Hilton*, Localization in topology, Amer. Math. Monthly 82 (1975), 113-131.

[3] *P. J. Hilton*, Editor, Localization in group theory and homotopy theory, LNM 418, Springer Berlin 1974.

[4] *H. Schubert*, Categories, Springer, Berlin 1972.

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P-ЛОКАЛИЗАЦИЈА ВО pro-Ab

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Резиме

Во оваа работа се докажани следните резултати,

1. За секоја абелова категорија A, pro-A е комплетна.

2. За секоја ко-комплетна абелова категорија A, pro-A е ко-комплетна.

3. За дадено множество P прости броеви и ко-комплетна абелова категорија A, дефиниран е функтор L_p: $A \to A$ со помош на бифункторот $- \otimes -$: $Ab \times A \to A$, каде Ab е категоријата абеловиц групи. Во случајот кога $A =$ pro-Ab, овој функтор врз A е обичниот функтор »P-локализација«. Меѓутоа, со пример е покажано дека тој не е еквивалентен со функторот »P-локализација на секое ниво«.