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ON INJECTIVE AND PROJECTIVE OBJECTS IN PRO-ABELIAN CATEGORIES

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Abstract. In this note the following two results are proved. 1. The existence of enough injectives in an abelian category implies the existence of enough injectives in the corresponding pro-category. 2. Zero is the only projective object in pro-(R-Mod).

The purpose of this note is to prove the following two theorems.

THEOREM 1. *If $\mathscr{A}$ is an abelian category with enough injectives, so is the category pro-$\mathscr{A}$.*

Let R be a ring with unit and R-Mod the category of left R-modules and homomorphisms.

THEOREM 2. *The only projective object in pro-(R-Mod) is zero.*

Note that results of Gabriel [3] imply that if $\mathscr{C}$ is a small abelian category with enough injectives then pro-$\mathscr{C}$, which is equivalent to $(\mathrm{Sex}(\mathscr{C}, \mathrm{Ab}))^{op}$, has enough projectives.

The reader is supposed to be familiar with the notion of pro--category, say to the extent of [1, Appendix]. A less readily available, but better reference for background to this note is [2].

Two things drew, in a different way, my attention to these questions: a claim by Porter (see for example [5], Fact 1. A., as well as the paragraph preceeding Theorem 6.1. in [2]) and a discussion with J. Vrabec, whom I wish to thank for it.

Proof of theorem 1. *If* Q is an injective object in $\mathscr{A}$, then Q, considered as an object in pro-$\mathscr{A}$ is injective. Indeed, any monomorphism $f\colon \mathbf{A} \to \mathbf{B}$ can be represented by level monomorphisms $f_i\colon A_i \to B_i$. Let $g\colon \mathbf{A} \to Q$ be represented by $g_j\colon A_j \to Q$. Since Q is injective in $\mathscr{A}$, there is a morphism $h_j\colon B_j \to Q$ in $\mathscr{A}$ such that $g_j = h_j f_j$. This h_j represents a morphism $h\colon \mathbf{B} \to Q$ in pro-$\mathscr{A}$ such that $g = hf$.

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Therefore, if Q_i, $i \in I$, is a family of injectives in $\mathscr{A}$, their product $*\Pi* Q_i$ in pro-$\mathscr{A}$ is injective (the product of a family of injectives in any category is again injective).

Now the remaining can be found in [2], Theorem 6.1.1.: if $\mathbf{A} = (A_i)_{i \in I}$ is any object in pro-$\mathscr{A}$, then $\mathbf{A}$ is the limit $*\lim* A_i$ in pro-$\mathscr{A}$ of the system of A_i's, over the directed set I. Now there is a canonical monomorphism

$$*\lim* A_i \to *\Pi* A_i.$$

If one chooses for each $i \in I$ a monomorphism $f_i: A_i \to Q_i$ in $\mathscr{A}$, where Q_i is injective, the composition of the obvious morphisms gives a monomorphism $\mathbf{A} \to *\Pi* Q_i$.

Proof of theorem 2. Let $\mathbf{A} = (A_i)_{i \in I}$ be any object of pro-(R-Mod). We shall construct an epimorphism $f: \mathbf{B} \to \mathbf{A}$ such that the only morphism from $\mathbf{A}$ to $\mathbf{B}$ is the zero morphism. This implies that if $\mathbf{A}$ is projective, it is zero.

Let J be a set whose cardinality is infinite and greater than

$$a = \Sigma \operatorname{card} A_i.$$

Let K be the set with objects the subsets of J whose complement has cardinality not greater than a if a is infinite, and finite if a is finite. For k, $k' \in K$, define $k > k'$ if $k \subset k'$. The set $I \times K$ is directed by $(i, k) > (i', k') \Leftrightarrow (i > i'$ and $k > k')$.

Consider the object $\mathbf{B} = (B_{(i,k)})_{(i,k) \in I \times K}$ where $B_{(i,k)}$ is the coproduct of copies of A_i, one for each element of k, and for each $(i, k) > (i', k')$ the bond $q: B_{(i,k)} \to B_{(i',k')}$ sends each copy of A_i to the corresponding copy of $A_{i'}$, by the bond p in A.

Let $f: \mathbf{A} \to \mathbf{B}$ be represented by $f_i: B_{(i,J)} \to A_i$, given on each summand A_i by the identity 1_{A_i}. To prove that f is an epimorphism, recall the criterion ([4], Theorem 5) for a morphism $h: \mathbf{C} = (C_m)_{m \in M} \to \mathbf{D} = (D_n)_{n \in N}$ in the category of pro-groups, represented by $h_n: C_{m(n)} \to D_n$, $n \in N$, to be epi: for each $n \in N$ and each $m > m(n)$, there should be an $n' > n$ such that $s(D_{n'}) \subset h_n r(C_m)$ where r and s denote the corresponding bonds in $\mathbf{C}$ and $\mathbf{D}$ respectively. Now for each $i \in I$ and bond $q: B_{(i',k)} \to B_{(i,J)}$ in $\mathbf{B}$, we have $f_i q(B_{(i',k)}) = p(A_{i'})$, where $p: A_{i'} \to A_i$ is the bond in $\mathbf{A}$; so f is epi.

Finally, we show that every morphism $g: \mathbf{A} \to \mathbf{B}$ is zero. Let g be represented by

$$g_{(i,k)}: A_{j(i,k)} \to B_{(i,k)}.$$

Then the image of each element of the R-module $A_{j(i,k)}$ is contained in a coproduct of a finite number of copies of A_i's, so by the choice of J and K there is a $k' > k$ such that the intersection of the image of

$g_{(i,k)}$ and the image of the bond $q: B_{(i,k')} \to B_{(i,k)}$ is $\{0\}$. On the other hand, by definition of a representative for a morphism in a pro-category, there is a $j' \in I$ with $j' \geq j(i,k), j(i,k')$ such that the diagram

$$\begin{array}{ccccc} & \nearrow & A_{j(i,k')} & \longrightarrow & B_{(i,k')} \\ A_{j'} & & & & \downarrow q \\ & \searrow^{p} & A_{j(i,k)} & \xrightarrow{g_{(i,k)}} & B_{(i,k)} \end{array}$$

commutes. Therefore $g_{(i,k)}\, p = 0$, i. e. g is the zero morphism.

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ЗА ИНЈЕКТИВНИТЕ И ПРОЕКТИВНИТЕ ОБЈЕКТИ ВО ПРО-АБЕЛОВИ КАТЕГОРИИ

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Резиме

Во оваа белешка се докажани следните два резултата:

1. Постоењето на доволно инјективни објекти во една абелова категорија $\mathscr{A}$ повлекува постоење на доволно инјективни во pro-$\mathscr{A}$.

2. Единствен проективен објект во категоријата pro-(R-Mod) каде R-Mod е категоријата леви модули над прстен со единица R, е нултиот.

(Забелешка: Резултати од Габриел [3] повлекуваат дека за мала абелова категорија $\mathscr{C}$ со доволно инјективни, pro-$\mathscr{C}$ има доволно проективни. Од резултатот 2. се гледа битноста на »малоста« на $\mathscr{C}$: проективните објекти во pro-$\mathscr{C}$ во тој случај се индексирани со усмерени множества чиј кардинален број е еднаков на кардиналниот број на Ob $\mathscr{C}$.)