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P-LOCALIZATION AND A TORSION THEORY IN pro-Ab

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Introduction

Let H denote the homotopy category of 1-connected CW complexes and let P be a set of primes. The theory of P-localization in H is well developed and understood (see for example [4]). If one wants to build a similar theory in the category pro-H (a reference for pro-categories is [1]), then one must first study P-localization in the category pro-Ab of pro-abelian groups. Two possibilities for a localization in pro-Ab come to mind. The first is to localize levelwise. This gives an exact functor, existence and explicit description are no problem, but it lacks the desired universal property: initiality with respect to morphisms into P-local pro-groups (i.e. those pro-groups $\boldsymbol{A}$ for which $p: \boldsymbol{A} \to \boldsymbol{A}$ is an isomorphism for each prime p not in P). The other possibility is to take this universal property as a starting point. But then one has to prove existence and loses an explicit description of the localization as well as, which is more important, exactness. It is this second approach that we follow in [8] and this paper.

For each set P of primes, we define a torsion theory $(\mathcal{T}_p, \mathcal{F}_p)$ in pro-Ab. We show that the class $\mathcal{T}_p$ of P-torsion pro-abelian groups is contained in the class of essentially P-torsion pro-abelian groups and that the P-localization is P-bijective.

1. Preliminaries

Let Ab denote the category of abelian groups. Recall [6, 17.7.6] that for any co-complete abelian category $\mathcal{A}$ there is a functor $-\otimes-: \mathrm{Ab} \times \mathcal{A} \to \mathcal{A}$, given by $(X, A) \mapsto X \otimes A$, and with the property that there is a natural, in $X \in |\mathrm{Ab}|$ and $A, B \in |\mathcal{A}|$, bijection

$$\theta : [X \otimes A, B]_{\mathcal{A}} \to [X, [A, B]_{\mathcal{A}}]_{\mathrm{Ab}} \tag{1}$$

Clearly, there is a natural isomorphism $A \to Z \otimes A$ in $\mathscr{A}$, where Z denotes the integers.

If P is a set of primes, let P' denote the set of primes not in P, and let Z_P be the subgroup of the rationals, Q, consisting of those fractions a/b with b a product of primes in P'.

An object A in $\mathscr{A}$ is said to be *P-local* if $p : A \to A$ (p times the identity) is an isomorphism for each $p \in P'$.

For any $A \in |\mathscr{A}|$, $A_P = Z_P \otimes A$ is P-local and the natural morphism $e : A \to A_P$ (induced from the inclusion $Z \to Z_P$) has the following universal property: any morphism $f : A \to B$, where B is P-local, factors uniquely through e [8, Theorem 3.4]. A morphism $L : A \to B$, with B P-local, having this universal property is called a *P-localization* of A.

An easy consequence of the adjunction (1) is the following lemma.

1.1. Lemma. (a) *For* $X \in |\mathrm{Ab}|$, *the functor* $X \otimes - : \mathscr{A} \to \mathscr{A}$ *commutes with colimits and is right exact.*

(b) *For* $A \in |\mathscr{A}|$, *the functor* $- \otimes A : \mathrm{Ab} \to \mathscr{A}$ *commutes with colimits and is right exact.*

The category pro-Ab of pro-abelian groups is co-complete [8], so all the above is valid in it.

2. A torsion theory in pro-Ab

2.1. Definitions. Following [2] a pair $(\mathscr{T}, \mathscr{F})$, where $\mathscr{T}$ and $\mathscr{F}$ are classes of objects in an abelian category $\mathscr{A}$, is called a *torsion theory* in $\mathscr{A}$ if the following conditions are satisfied:

(i) $\mathscr{T} \cap \mathscr{F} = \{0\}$.

(ii) If $T \to A \to 0$ is exact with $T \in \mathscr{T}$, then $A \in \mathscr{T}$.

(iii) If $0 \to A \to F$ is exact with $F \in \mathscr{F}$, then $A \in \mathscr{F}$.

(iv) For each $A \in |\mathscr{A}|$ there is an exact sequence $0 \to T \to A \to F \to 0$ with $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

We shall now define a torsion theory in pro-Ab for each set P of primes.

Call an object $\mathbf{A}$ in pro-Ab *P-torsion* (or simply *torsion* if $P =$ all primes) if $\mathbf{A}_{P'} = 0$. An object $\mathbf{A}$ is *P-torsion free* if the localization $e : \mathbf{A} \to \mathbf{A}_{P'}$ is a monomorphism. Let $\mathscr{T}_P$ denote the class of P-torsion objects in pro-Ab and $\mathscr{F}_P$ the class of P-torsion free objects in pro-Ab.

Clearly, axioms (i), (ii) and (iii) for a torsion theory are satisfied.

2.2. Proposition. *If* $\mathbf{A} \in |\text{pro-Ab}|$, *the* P'*-localization* $e : \mathbf{A} \to \mathbf{A}_{P'}$ *is such that* $\mathbf{A}/\mathrm{Ker}\ e$ *is P-torsion free.*

Proof. The exact sequence

$$0 \to \operatorname{Ker} e \to \boldsymbol{A} \to \boldsymbol{A}/\operatorname{Ker} e \to 0$$

implies the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} e \arrow[r, "f"] \arrow[d] & \boldsymbol{A} \arrow[r] \arrow[d, "e"] & \boldsymbol{A}/\operatorname{Ker} e \arrow[r] \arrow[d] & 0 \\
 & (\operatorname{Ker} e)_{P'} \arrow[r] & \boldsymbol{A}_{P'} \arrow[r] & (\boldsymbol{A}/\operatorname{Ker} e)_{P'} \arrow[r] & 0.
\end{tikzcd}$$

The first morphism in the second row is zero because the composite ef is zero and by the universal property of $\operatorname{Ker} e \to (\operatorname{Ker} e)_{P'}$. Since $0 \to \boldsymbol{A}/\operatorname{Ker} e \xrightarrow{g} \boldsymbol{A}_{P'}$ is exact, the commutative square

$$\begin{tikzcd}
\boldsymbol{A} \arrow[r] \arrow[d] & \boldsymbol{A}/\operatorname{Ker} e \arrow[d] \arrow[dl, "g"'] \\
\boldsymbol{A}_{P'} \arrow[r, "\cong"'] & (\boldsymbol{A}/\operatorname{Ker} e)_{P'}
\end{tikzcd}$$

implies that $\boldsymbol{A}/\operatorname{Ker} e \to (\boldsymbol{A}/\operatorname{Ker} e)_{P'}$ is mono.

We shall prove in Section 3 that for any $\boldsymbol{A} \in |\text{pro-Ab}|$, $\operatorname{Ker} e$ is P-torsion, so that the pair $(\mathscr{T}_P, \mathscr{F}_P)$ satisfies also axiom (iv) for a torsion theory.

This torsion theory is not hereditary, i.e. a subobject of a torsion object need not be torsion, as the following example shows.

2.3. Example. Let $\boldsymbol{B} = \bigoplus Z/pZ$, the sum being over all primes p (i.e. $\boldsymbol{B}$ is indexed by the one-point category). Order the set of all primes: $p_1, p_2, \ldots$. Let $\boldsymbol{A}$ be

$$\bigoplus_{p \neq p_1} Z/pZ \leftarrow \bigoplus_{p \neq p_1, p_2} Z/pZ \leftarrow \cdots$$

with inclusions as bonding morphisms. Then clearly $\boldsymbol{A}$ is a subobject of $\boldsymbol{B}$ in pro-Ab, the monomorphism being given by the inclusions

$$\begin{tikzcd}
\bigoplus_{p \neq p_1} Z/pZ \arrow[d, hook] & \bigoplus_{p \neq p_1, p_2} Z/pZ \arrow[l] \arrow[d, hook] & \cdots \arrow[l] \\
\bigoplus Z/pZ & \bigoplus Z/pZ \arrow[l, "\mathrm{Id}"'] & \cdots \arrow[l, "\mathrm{Id}"']
\end{tikzcd}$$

Further, $Q \otimes \boldsymbol{B} = 0$ since $\bigoplus Z/pZ$ is a torsion albelian group and $Q \otimes -$ restricts on Ab to the usual tensor product [8].

In order to show that $Q \otimes \boldsymbol{A} \neq 0$, note that $\boldsymbol{A} \neq 0$ and that $\boldsymbol{A}$ is 0-local, i.e. $p: \boldsymbol{A} \to \boldsymbol{A}$ is an isomorphism for each prime p, so that $Q \otimes \boldsymbol{A} \cong \boldsymbol{A}$ (see Section 1).

Following [7] define an object $\boldsymbol{A} = (A_i)_{i\in I}$ in pro-Ab to be *essentially P-torsion* if $\boldsymbol{A} \cong \boldsymbol{B} = (B_j)_{j\in J}$ with each B_j a P-torsion abelian group. This is equivalent (see [3, Proposition 2.1]) to the following: for each $i \in I$ there is an $i' \geq i$ such that $\alpha_{i'i}(A_{i'})$ is a P-torsion abelian group (here α denotes the appropriate bonding map in $\boldsymbol{A}$).

2.4. Proposition. *If $\boldsymbol{A}$ is in $\mathcal{T}_P$, then $\boldsymbol{A}$ is essentially P-torsion.*

Proof. If $\boldsymbol{A} = (A_i)_{i\in I}$ is such that $Z_{P'} \otimes \boldsymbol{A} = 0$, then each $\boldsymbol{A} \to \boldsymbol{B}$, with $\boldsymbol{B}$ P'-local, is zero. Take $\boldsymbol{B} = (Z_{P'} \otimes A_i)_{i\in I}$. Then $\boldsymbol{B}$ is obviously P'-local, since for each $p \in P$ the morphism $p: Z_{P'} \otimes A_i \to Z_{P'} \otimes A_i$ is an isomorphism. Therefore, $f: \boldsymbol{A} \to \boldsymbol{B}$, given by $f_i: A_i \to Z_{P'} \otimes A_i$, is zero. This means that for each $i \in I$, there is an $i' \geq i$ in I, such that $f_i \alpha_{i'i} = 0$, i.e. $\alpha_{i'i}(A_{i'})$ is a subgroup of A_i which tensored with $Z_{P'}$ is zero, i.e. is a P-torsion abelian group.

2.5. Remark. It is not true that every essentially P-torsion pro-abelian group is in $\mathcal{T}_p$: the pro-group $\boldsymbol{A}$ in Example 2.3 provides a counter-example. One can even construct an example of such a pro-group $\boldsymbol{C} = (C_n)_{n\in N}$, with C_n a finite group for each natural number n.

2.6. Example. Let p be a prime and let C be

$$Z/pZ \leftarrow Z/p^2Z \leftarrow Z/p^3Z \leftarrow \cdots$$

with canonical projections as bonding morphisms. Let $P' = \{p\}$. To show that $Z_P \otimes \boldsymbol{C} \neq 0$, we show that there is a non-zero element in $[Z_P \otimes \boldsymbol{C}, \boldsymbol{B}]_{\text{pro-Ab}}$ for some pro-group $\boldsymbol{B}$, or equivalently, that there is a non-zero element in $[Z_P, [\boldsymbol{C}, \boldsymbol{B}]_{\text{pro-Ab}}]_{\text{Ab}}$.

Let $\boldsymbol{B} = Q/Z$ (indexed by the one-point category) and $f: \boldsymbol{C} \to \boldsymbol{B}$ be generated by $Z/pZ \to Q/Z$ defined by $1 \mapsto 1/p$. This f is divisible in $[\boldsymbol{C}, \boldsymbol{B}]_{\text{pro-Ab}}$ by any power of p, so there is a non-zero homomorphism $Z_P \to [\boldsymbol{C}, \boldsymbol{B}]_{\text{pro-Ab}}$ given by $1 \mapsto f$.

2.7. Remark. In pro-Ab there is not a decomposition of torsion objects as a direct sum, over all primes p, of p-rimary components. For example, let $p_1, p_2, \ldots$ be an ordering of the primes, $\boldsymbol{A}_n = Z/p_nZ$, indexed by the one-point category. Then in pro-Ab, $\bigoplus \boldsymbol{A}_n$ is $\bigoplus Z/p_nZ$, indexed by the one-point category. and $\Pi\, \boldsymbol{A}_n$ is isomorphic to

$$Z/p_1Z \leftarrow (Z/p_1Z) \otimes (Z/p_2Z) \leftarrow \cdots$$

The canonical morphism

$$h: \bigoplus \boldsymbol{A}_n \to \prod \boldsymbol{A}_n$$

is an epimosphism but not a monomorphism: its kernel is the pro-group $\boldsymbol{A}$ from Example 2.3. So, $\prod \boldsymbol{A}_n$ is torsion, and $(\prod \boldsymbol{A}_n)_{p_n} \cong \boldsymbol{A}_n$.

Note that the fact that h is not a monomorphism shows that $(AB5)$ in an abelian category C does not necessarily imply $(AB5)$ in pro-C (cf. [6, 14.6.8]).

2.8. Lemma. *If $\boldsymbol{A}$ is P-torsion, then $\boldsymbol{A} \to \boldsymbol{A}_P$ is an isomorphism.*

Proof. Since $\boldsymbol{A}$ is P-torsion, it is essentially P-torsion; so for any $p \in P'$, $p: \boldsymbol{A} \to \boldsymbol{A}$ is an isomorphism.

2.9. Lemma. *If $\boldsymbol{A}$ is torsion and essentially P-torsion, then $\boldsymbol{A}$ is P-torsion.*

Proof. Tensoring the exact sequence $Z \to Z_{P'} \oplus Z_P \to Q$ with $\boldsymbol{A}$ we obtain $\boldsymbol{A} \to \boldsymbol{A}_{P'} \oplus \boldsymbol{A}_P \to 0$. Since $\boldsymbol{A}$ is essentially P-torsion, $\boldsymbol{A} \to \boldsymbol{A}_P$ is an isomorphism. Therefore $\boldsymbol{A} \to \boldsymbol{A}_{P'} \oplus \boldsymbol{A}_P$ is also a monomorphism, i.e. $\boldsymbol{A}_{P'} = 0$.

2.10. Lemma. *If $\boldsymbol{A}$ is torsion, then $\boldsymbol{A} \to \boldsymbol{A}_{P'} \oplus \boldsymbol{A}_P$ is an isomorphism.*

Proof. Since $\boldsymbol{A} = (A_i)_{i \in I}$ is essentially torsion, we may suppose without loss of generality that each A_i is a torsion abelian group. So, $A_i \cong A_i^1 \oplus A_i^2$ where A_i^1 is P'-torsion and A_i^2 is P-torsion. Moreover, the bonding maps clearly send A_j^1 to A_i^1 and A_j^2 to A_i^2. So $\boldsymbol{A} \cong \boldsymbol{A}_1 \oplus \boldsymbol{A}_2$, where $\boldsymbol{A}_1 = (A_i^1)_{i \in I}$ is essentially P'-torsion and $\boldsymbol{A}_2 = (A_i^2)_{i \in I}$ is essentially P-torsion. Further, both are torsion for they are quotients of the torsion object $\boldsymbol{A}$. So, by Lemma 2.9 and Lemma 2.8, $\boldsymbol{A}_1 \cong \boldsymbol{A}_{P'}$ and $\boldsymbol{A}_2 \cong \boldsymbol{A}_P$.

2.11. Proposition. *Let $\boldsymbol{A} \in |\text{pro-Ab}|$. Then $\boldsymbol{A} = 0$ if and only if for each prime p, $\boldsymbol{A}_p = 0$.*

Proof. We show the non-obvious implication.

If $\boldsymbol{A}_p = 0$, then $Q \otimes \boldsymbol{A} = 0$, i.e. A is torsion. For each prime p, from $0 \to \boldsymbol{A} \to \boldsymbol{A}_p \oplus \boldsymbol{A}_{p'} \to 0$ we get that $\boldsymbol{A}/\boldsymbol{A}_p$ is such that p times its identity is an isomorphism. Also, p times the identity of $\bigoplus_{q \neq p} \boldsymbol{A}_q$ is an isomorphism.

Let $f: \bigoplus \boldsymbol{A}_p \to \boldsymbol{A}$ be induced from $\boldsymbol{A}_p \to \boldsymbol{A}$ and let $\boldsymbol{B} = \operatorname{Coker} f$. For each prime p there is an exact sequence

$$\bigoplus_{q \neq p} \boldsymbol{A}_q \to \boldsymbol{A}/\boldsymbol{A}_p \to \boldsymbol{B} \to 0$$

(this is the Coker sequence obtained from

$$\begin{tikzcd}
\boldsymbol{A}_p \arrow[r] \arrow[d] & \boldsymbol{A}_p \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
\oplus \boldsymbol{A}_q \arrow[r] & \boldsymbol{A} \arrow[r] & \boldsymbol{B} \arrow[r] & 0).
\end{tikzcd}$$

So, $p: \boldsymbol{B} \to \boldsymbol{B}$ is an isomorphism for each prime p. i.e. $Q \otimes \boldsymbol{B} \cong \boldsymbol{B}$. But $Q \otimes \boldsymbol{B} = 0$ since $\boldsymbol{B}$ is a factor object of a torsion object. This means that $\boldsymbol{B} = 0$, i.e. f is an epimorphism. Therefore $\boldsymbol{A}_p = 0$ for each prime p implies $\boldsymbol{A} = 0$.

2.12. Corollary. *A morphism $h: \boldsymbol{A} \to \boldsymbol{B}$ in* pro-Ab *is an epimorphism if and only if for each prime p, $h_p: \boldsymbol{A}_p \to \boldsymbol{B}_p$ is an epimorphism.*

3. Torsion products in pro-Ab

3.1. Proposition. *Let $\boldsymbol{A} \in |$*pro-Ab$|$. Then the functor $-\otimes \boldsymbol{A}$: Ab $\to$ pro-Ab.

(a) *has left satellites $tor^n(-, \boldsymbol{A})$*

(b) *for $n \geqslant 2$,* $\operatorname{tor}^n(-, \boldsymbol{A}) = 0$.

Proof. (a) follows from the existence of enough projectives in Ab. The ring Z is a principal ideal domain, hence (b).

Denote $\operatorname{tor}^1(X, \boldsymbol{A})$ by $X*\boldsymbol{A}$.

So, if $0 \to X' \to X \to X'' \to 0$ is an exact sequence of abelian groups, the sequence

$$0 \to X'*\boldsymbol{A} \to X*\boldsymbol{A} \to X''*\boldsymbol{A} \to X' \otimes \boldsymbol{A} \to X \otimes \boldsymbol{A} - X'' \otimes \boldsymbol{A} \to 0$$

is exact for any $\boldsymbol{A} \in |$pro-Ab$|$.

3.2. Lemma. *If X is a free abelian group, $0 \to \boldsymbol{A} \to \boldsymbol{B} \to \boldsymbol{C} \to 0$ is an exact sequence in* pro-Ab, *then $0 \to X \otimes \boldsymbol{A} \to X \otimes \boldsymbol{B} \to X \otimes \boldsymbol{C} \to 0$ is exact.*

Proof. We can suppose [5, Proposition 2.4] that $0 \to \boldsymbol{A} \to \boldsymbol{B} \to \boldsymbol{C} \to 0$ consists of level exact sequences of abelian groups. Further, for $\boldsymbol{D} \in |$pro-Ab$|$, $X \otimes \boldsymbol{D}$ is isomorphic to a coproduct of copies of $\boldsymbol{D}$, one for each generator of X. Therefore $0 \to X \otimes \boldsymbol{A} \to X \otimes \boldsymbol{B} \to X \otimes \boldsymbol{C} \to 0$ consists of level exact sequences of abelian groups, so is exact in pro-Ab.

3.3. Proposition. *Let $0 \to \boldsymbol{A} \to \boldsymbol{B} \to \boldsymbol{C} \to 0$ be an exact sequence in* pro-Ab, *X an abelian group. Then the sequence*

$$0 \to X*\boldsymbol{A} \to X*\boldsymbol{B} \to X*\boldsymbol{C} \to X \otimes \boldsymbol{A} \to X \otimes \boldsymbol{B} \to X \otimes \boldsymbol{C} \to 0$$

is exact.

Proof. Let $0 \to R \to F \to X \to 0$ be a free resolution of X. The Ker-Coker sequence obtained from

$$\begin{tikzcd}
0 \arrow[r] & R \otimes \boldsymbol{A} \arrow[r] \arrow[d] & R \otimes \boldsymbol{B} \arrow[r] \arrow[d] & R \otimes \boldsymbol{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \otimes \boldsymbol{A} \arrow[r] & F \otimes \boldsymbol{B} \arrow[r] & F \otimes \boldsymbol{C} \arrow[r] & 0
\end{tikzcd}$$

implies the result.

3.4. Lemma. *If X and Y are abelian groups such that their tensor product $X \otimes Y = 0$ and their torsion product $X * Y = 0$ and $\boldsymbol{A}$ is a pro-abelian group, then $X*(Y \otimes \boldsymbol{A}) = 0$.*

Proof. Let

$$0 \to R \to F \to X \to 0 \tag{2}$$

be a free resolution of X. The hypothesis of the lemma imply that $0 \to R \otimes Y \to F \otimes Y \to 0$ is exact. Tensoring (2) with $Y \otimes \boldsymbol{A}$ we obtain

$$0 \to X*(Y \otimes \boldsymbol{A}) \to R \otimes (Y \otimes \boldsymbol{A}) \to F \otimes (Y \otimes \boldsymbol{A}) \to 0$$

and hence the result.

3.5. Lemma. *Let X and Y be abelian groups, Y free and $\boldsymbol{A}$ a pro-abelian group. Then $X*(Y \otimes \boldsymbol{A}) \cong Y \otimes (X * \boldsymbol{A})$ in a natural way.*

Proof. In the notation of the previous lemma, the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \otimes (X * \boldsymbol{A}) \arrow[r] \arrow[d, "f"] & Y \otimes R \otimes \boldsymbol{A} \arrow[r] \arrow[d, "\cong"] & Y \otimes F \otimes \boldsymbol{A} \arrow[d, "\cong"] \\
0 \arrow[r] & X * (Y \otimes \boldsymbol{A}) \arrow[r] & R \otimes Y \otimes \boldsymbol{A} \arrow[r] & F \otimes Y \otimes \boldsymbol{A}
\end{tikzcd}$$

has exact rows, so f is an isomorphism.

Finally, we prove the following proposition.

3.6. Proposition. *Let $e: \boldsymbol{A} \to \boldsymbol{A}_{P'}$ be the P'-localization of $\boldsymbol{A} \in |\text{pro-Ab}|$. Then* Ker *$e$ is P-torsion.*

Proof. (I wish to thank Dr. B. Gray for this idea of the proof.) To shorten notation, we only consider the case $P =$ all primes.

The exact sequence $0 \to Z \to Q \to Q/Z \to 0$ induces

$$0 \to Q*\boldsymbol{A} \to (Q/Z)*\boldsymbol{A} \to \boldsymbol{A} \to Q \otimes \boldsymbol{A} \to (Q/Z) \otimes \boldsymbol{A} \to 0.$$

Since there is an epimorphism $(Q/Z)*\boldsymbol{A} \to \operatorname{Ker} e$, it is enough to prove that $Q \otimes [(Q/Z)*\boldsymbol{A}] = 0$.
Let

$$0 \to R \to F \to Q/Z \to O \tag{3}$$

be a free resolution of Q/Z. Tensoring it with $\boldsymbol{A}$, we get

$$0 \to (Q/Z)*\boldsymbol{A} \to R \otimes \boldsymbol{A} \to F \otimes \boldsymbol{A} \to (Q/Z) \otimes \boldsymbol{A} \to 0. \tag{4}$$

This splits into two exact sequences

$$0 \to (Q/Z)*\boldsymbol{A} \to R \otimes \boldsymbol{A} \to \boldsymbol{I} \to 0,$$

$$0 \to \boldsymbol{I} \to F \otimes \boldsymbol{A} \to (Q/Z) \otimes \boldsymbol{A} \to 0. \tag{5}$$

Tensoring (5) with Q, we get

$$0 \to Q*[(Q/Z)*\boldsymbol{A}] \to Q*(R\otimes \boldsymbol{A}) \to Q*\boldsymbol{I} \to Q\otimes[(Q/Z)*\boldsymbol{A}] \to$$

$$\to Q\otimes R\otimes \boldsymbol{A} \to Q\otimes \boldsymbol{I} \to 0,$$

$$0 \to Q*\boldsymbol{I} \to Q*(F\otimes \boldsymbol{A}) \to Q*[(Q/Z)\otimes \boldsymbol{A}] \to Q\otimes \boldsymbol{I} \to$$

$$\to Q\otimes F\otimes \boldsymbol{A} \to Q\otimes(Q/Z)\otimes \boldsymbol{A} \to 0.$$

Since $Q*[(Q/Z)\otimes\boldsymbol{A}]=0$ by Lemma 3.4 and since $Q\otimes(Q/Z)\otimes\boldsymbol{A}=0$, these exact sequences combine to give the exact sequence

$$0 \to Q*[(Q/Z)*\boldsymbol{A}] \to Q*(R\otimes \boldsymbol{A}) \xrightarrow{f} Q*(F\otimes \boldsymbol{A}) \to Q\otimes[(Q/Z)*\boldsymbol{A}] \to$$

$$\to Q\otimes R\otimes \boldsymbol{A} \xrightarrow{g} Q\otimes F\otimes \boldsymbol{A} \to 0$$

One easily sees that g is an isomorphism, so Coker $f \cong Q\otimes[(Q/Z)*\boldsymbol{A}]$. By Lemma 3.5

$$Q*(F\otimes \boldsymbol{A}) \cong F\otimes(Q*\boldsymbol{A}) \quad \text{and} \quad Q*(R\otimes \boldsymbol{A}) \cong R\otimes(Q*\boldsymbol{A}).$$

Tensoring (3) with $Q * \boldsymbol{A}$ we get

$$0 \to (Q/Z)*(Q*\boldsymbol{A}) \to R\otimes(Q*\boldsymbol{A}) \to F\otimes(Q*\boldsymbol{A}) \to (Q/Z)\otimes(Q*\boldsymbol{A}) \to 0.$$

So Coker $f \cong (Q/Z)\otimes(Q*\boldsymbol{A})$ and therefore $Q\otimes[(Q/Z)*\boldsymbol{A}] \cong (Q/Z)\otimes(Q*\boldsymbol{A})$. But the left hand side, being local, is torsion free, the right hand side is torsion, so $Q\otimes[(Q/Z)*\boldsymbol{A}]=0$.

It follows from Proposition 3.6 that $e: \boldsymbol{A} \to \boldsymbol{A}_{P'}$ is P'-bijective, i.e. Ker e and Coker e are P-torsion pro-groups. I do not know as yet whether this characterizes the P'-localization, that is whether the following is true: If $0 \to \boldsymbol{T}_1 \to \boldsymbol{A} \to \boldsymbol{B} \to \boldsymbol{T}_2 \to 0$ is exact, $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are P-torsion and $\boldsymbol{B}$ is P'-local, then $\boldsymbol{B} \cong \boldsymbol{A}_{P'}$.

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