SEMIGROUPS IN WHICH SOME LEFT IDEAL IS A GROUP

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In this note we give a structural theorem for semigroups containing a left ideal which is a group. This theorem is a generalization of the main results of the papers [2], [3] and [4]

Theorem. Some left ideal of a semigroup S is a group if and only if S is isomorphic to a semigroup $G\times JUP=[G, J, P, \varphi, \xi]$, where

(i) G is a group, P is a partial semigroup[1]) (which may be empty), J is a non-empty set and $G\times J\cap P=\varnothing$;

(ii) $\varphi: p\to p\varphi$ is a homomorphism from P into G and $\xi: p\to\xi_p$ a homomorphism from P into T_J[2]) such that $\xi_p\,\xi_q$ is a constant if the product pq is not defined in P;

(iii) the product of two elements of the set $G\times J\,U\,P$ is defined by:

(1) $(x, i)(y, j)=(xy, j)$, (2) $(x, i)p=(xp\varphi, i\xi_p)$, (3) $p(x, i)=(p\varphi x, i)$,

(4) $pq=r$ in $P\Rightarrow pq=r$ in $G\times J\,U\,P$, and (5) $pq\in P\Rightarrow pq=(p\varphi q\varphi, i\xi_p\,\xi_q)$, where $x, y\in G$, $p, q, r\in P$, $i, j\in J$.

Proof. If G, J, P, φ and ξ satisfy the conditions (i) and (ii) and if an operation is defined in $G\times J\,U\,P$ by (iii), then it can be easily seen that $G\times J\,U\,P=[G, J, P;\ \varphi, \xi]$ is a semigroup and if $G_i=\{(x, i); x\in G\}$ then $\{G_i; i\in J\}$ is a collection of left ideals which are groups isomorphic to G.

Suppose now that S is a semigroup in which some left ideal G_i is a group. Then G_i is a minimal left ideal and therefore (see, for example, [1]) $\{G_is; s\in S\}$ is the collection of all minimal left ideals of S.

The element s may not belong to G_is but we can chose an $r\in G_is$ such that $r\in G_i r=G_is$; thus we may assume $s\in G_is$ and then $s=us$ for some $u\in G_i$. Let $x, y\in G_is$ and $x=x_1 s$, $y=y_1 s$ where $x_1, y_1\in G_i$; if e is the identity element of G_i then $se\in G_i$ and $se=x_1^{-1}\,y_1\,u\,z_1^{-1}$ for some $z_1\in G_i$. From $s=us$ it follows $xz=y$ where $z=z_1\,s\in G_is$. Thus $x\,G_is=G_is$ for every $x\in G_is$, i. e. G_is is a right simple semigroup As a minimal left ideal G_is is also a left simple semigroup, and therefore it is a group. Thus we have proved that every minimal left ideal of S is a group.

Let $\{G_i;\ i\in J\}$ be the collection of all (different) minimal left ideals of the semigroup S and let $K=\bigcup_{i\in J} G_i$. Then K is a two-sided ideal of S and $\{G_i;\ i\in J\}$ is a collection of left ideals of K which are groups. Therefore (see, for example, [5] Lemma 2) there is a group G such that K is isomorphic to a semigroup $G\times J$ where the product is defined by (1); and we assume that $K=G\times J$. The subset $P=S\setminus K$ is a partial semigroup and we assume it to be non-empty.

Let $p\in P$, $i, j\in J$ be arbitrary elements and e the identity element of G. Then

$$p(e, i)=((p, i)\varphi, i)\in G_i=\{(x, i); x\in G\}\,^{3)}$$

(because G_i is a left ideal of S) and

$$((p, i)\varphi, i)=p(e, i)=p(e, j)(e, i)=((p, j)\varphi, j)(e, i)=((p, j)\varphi, i),$$

i. e. $(p, i)\varphi$ does not depend on i; and so

(6) $p(e, i)=(p\varphi, i)$,

whepe φ is a mapping of P into G. Also

$$(e, i)\,p=((i, p)\,\psi, i\xi_p)\in G\times J^{3)},$$

[1]) P is a partial semigroup if a partial binary operation is defined in P such that $p\cdot qr\in P \Longleftrightarrow pq\cdot r\in P$ and then $p\cdot qr=pq\cdot r$.

[2]) T_J is the semigroup of all mappings from J into itself.

[3]) $(p, i)\varphi$, $(i, p)\psi\in G$, $i\xi_p\in J$

for $G \times J$ is a right ideal of S. Then

$$((i,p)\psi, i) = ((i,p)\psi, i\xi_p)(e,i) = (e,i)p(e,i) = (p\varphi, i),$$

i. e. $(i,p)\psi = p\varphi$ for every $i \in J$, $p \in P$; thus

(7) $(e,i)p = (p\varphi, i\xi_p)$

where $\xi : p \to \xi_p$ is a mapping of P into T_J. From (6) and (7) there follow

$$p(x,i) = p(e,i)(x,i) = (p\varphi, i)(x,i) = (p\varphi x, i)$$

and

$$(x,i)p = (x,i)(e,i)p = (x,i)(p\varphi, i\xi_p) = (xp\varphi, i\xi_p),$$

i. e. the equations (2) and (3) are satisfied.

Let $p, q \in P$. If $pq = r \in P$ then

$$(r\varphi, i\xi_r) = (e,i)r = (e,i)pq = (p\varphi q\varphi, i\xi_p\xi_q),$$

whence follows that φ and ξ are homomorphisms. If $pq = (x,k) \in G \times J$, then, for every $i \in J$,

$$(x,k) = (e,i)(x,k) = (e,i)pq = (p\varphi q\varphi, i\xi_p\xi_q),$$

whence follows the equation (5) and that $\xi_p \xi_q$ is a constant. This completes the proof of Theorem.

Some notes.

1. For an arbitrary group G, a non-empty set J and a partial semigroup P, there exists a $[G, J, P; \varphi, \xi]$ -semigroup. For example, we can put $p\varphi = e$, $i\xi_p = k$ where k is a fixed element of J; if $P = \varnothing$ then $[G, J, P; \varphi, \xi] = G \times J$.

2. Two semigroups $[G, J, P; \varphi, \xi]$ and $[G^*, J^*, P^*; \varphi^*, \xi^*]$ are isomorphic if and only if there exist an isomorfism α from G onto G^*, an isomorphism β from P onto P^* and a one-to-one mapping γ from J onto J^*, such that $\beta\varphi^* = \varphi\alpha$ and $\xi_p \gamma = \gamma \xi^*_{p\beta}$.

3. $G \times J$ is the minimal two-sided ideal and also the unique minimal right ideal of a $[G, J, P; \varphi, \xi]$ -semigroup; therefore it is the set of all right zeroid elements of the semigroup [1]. From this follows that if a semigroup contains a left ideal and a right ideal which are groups then the semigroup contains two-sided zeroid elements.

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ПОЛУГРУПИ ВО КОИ НЕКОЈ ИДЕАЛ Е ГРУПА

Резиме

Во работата е докажан следниот резултат. Некој лев идеал на полугрупата S е група ако и само ако S е изоморфна со некоја полугрупа од облик $G \times J \cup P = [G, J, P; \varphi, \xi]$ каде: (i) G е група, J е напразно множество, а P е делумична полугрупа дисјунктна со множеството $G \times J$ (P може да биде и празно множество); (ii) φ: $p \to p\varphi$ е хомоморфизам од P во G, а ξ: $p \to \xi p$ е хомоморфизам од P во T_J таков да $\xi_p \xi_q$ е константна ако производот $p\,q$ не е определен во P; (iii) операцијата во множеството $G \times J \cup P$ е определена со (1)—(5).

При тоа под делумична полугрупа подразбираме алгебарска структура со една делумична бинарна операција која е асоцијативна, т.е. $pq \cdot r \in P \iff p \cdot qr \in P$ и при тоа $pq \cdot r = p \cdot qr$. Со T_J ја означуваме полугрупата од сите трансформации на множеството J, т.е. пресликувања од J во J; операцијата во таа полугрупа е обичното множење на пресликувања.