POLYNOMIAL SUBALGEBRAS

Algebraic Conference, Novi Sad 1981; (1982), 1–6

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A polynomial subalgebra of an algebra $\underline{A} = (A, \mathcal{O})$ is a subset B of the carrier of the algebra which is closed under the polynomials belonging to a set of $\mathcal{O}$-polynomials. In this paper polynomial subalgebras are considered, together with a few properties and examples. A special attention is given to the polynomial subalgebras of the algebras belonging to a variety.

1. Throughout the paper $\mathcal{O}$ and $\mathcal{O}'$ will be two sets of operational symbols and $X = \{x_1, x_2, \dots, x_n, \dots\}$ will be the set of individual variables. By $\mathcal{O}_X$ will be denoted the set of all $\mathcal{O}$-polynomials, i.e. $\mathcal{O}_X$ = Term ($\mathcal{O}$). If $p \in \mathcal{O}_X$ and if each variable that occurs in p is in the set $\{x_1, \dots, x_n\}$, then we will usually write $p = p(x_1, \dots, x_n)$. Let $\wedge : \mathcal{O} \to \mathcal{O}'_X$ be a mapping such that if $f \in \mathcal{O}(n)$, then $f^\wedge = f^\wedge(x_1, \dots, x_n)$. The mapping $\wedge$ induces a mapping from $\mathcal{O}_X$ into $\mathcal{O}'_X$ (denoted with the same symbol $\wedge$) defined by: (i) $x^\wedge = x$, for each $x \in X$ and (ii) $f \in \mathcal{O}(n)$, $p = fp_1 \dots p_n \Rightarrow p^\wedge = f^\wedge(p_1^\wedge, \dots, p_n^\wedge)$.

Let $\underline{A}$ be an $\mathcal{O}$-algebra, $\underline{A}'$ an $\mathcal{O}'$-algebra and $\phi : A \to A'$ a mapping such that $\phi(f_{\underline{A}}(a_1, \dots, a_n)) = f^\wedge_{\underline{A}'}(\phi(a_1), \dots, \phi(a_n))$ for each $f \in \mathcal{O}(n)$ and $a_1, \dots, a_n \in A$. The mapping ϕ in this case will be called a $\wedge$-homomorphism from $\underline{A}$ into $\underline{A}'$. Moreover, if $A \subseteq A'$ and if the embedding of $\underline{A}$ into $\underline{A}'$ is a $\wedge$-homomorphism, then $\underline{A}$ is said to be a $\wedge$-subalgebra of $\underline{A}'$. (We will sometimes say polynomial homomorphism (polynomial subalgebra) instead of $\wedge$-homomorphism ($\wedge$-subalgebra).)

If $\underline{A}'$ is an $\mathcal{O}'$-algebra, then an $\mathcal{O}$-algebra $\underline{{}^\wedge A'}$ by the same carrier A' is defined by: $f_{{}^\wedge\underline{A}'}(a_1', \dots, a_n') = f^\wedge_{\underline{A}'}(a_1', \dots, a_n')$, for each $f \in \mathcal{O}(n)$ and $a_1', \dots, a_n' \in A'$. We say that $\underline{{}^\wedge A'}$ is induced from $\underline{A}'$ by $\wedge$.

Let $\mathcal{C}'$ be a class of $\mathcal{O}'$-algebras and $\mathcal{C}$ be a class of $\mathcal{O}$-algebras. Then by ${}^\wedge\mathcal{C}'$ will be denoted the class of $\mathcal{O}$-algebras which are $\wedge$-subalgebras of $\mathcal{O}'$-algebras belonging to $\mathcal{C}'$, and by $\mathcal{C}^\wedge$ the class of $\mathcal{O}'$-algebras $\underline{A}'$ such that all $\wedge$-subalgebras of $\underline{A}'$ are in $\mathcal{C}$. We say that a pair $(\mathcal{C}, \mathcal{C}')$ is $\wedge$-compatible if each algebra $\underline{A} \in \mathcal{C}$ is a $\wedge$-subalgebra of an algebra $\underline{A}' \in \mathcal{C}'$ such that ${}^\wedge\underline{A}' \in \mathcal{C}$.

The following properties give some connections between $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}^\wedge$ and ${}^\wedge\mathcal{C}'$.

$\underline{1}^\circ$. (a) If $\mathcal{C}$ is a class of $\mathcal{O}$-algebras and $\mathcal{C}'$ a class of $\mathcal{O}'$-algebras, then: ${}^\wedge(\mathcal{C}^\wedge) \subseteq \mathcal{C}$, $\quad \mathcal{C}' \subseteq ({}^\wedge\mathcal{C}')^\wedge$.

(b) The equation ${}^\wedge(\mathcal{C}^\wedge) = \mathcal{C}$ holds iff each $\mathcal{O}$-algebra $\underline{A} \in \mathcal{C}$ is a $\wedge$-subalgebra of an $\mathcal{O}'$-algebra $\underline{A}'$ such that each $\wedge$-subalgebra of $\underline{A}'$ is in $\mathcal{C}$.

(c) The equation $(^\wedge\mathcal{C}')^\wedge = \mathcal{C}'$ holds iff $\mathcal{C}'$ contains any σ'-algebra $\underline{A}'$ such that every $\wedge$-subalgebra $\underline{A}$ of $\underline{A}'$ is $\wedge$-subalgebra of $A'' \in \mathcal{C}'$.

$\underline{2}^o$. If $(\mathcal{C}, \mathcal{C}')$ is a $\wedge$-compatible, then $\mathcal{C} \subseteq {}^\wedge\mathcal{C}'$.

$\underline{3}^o$. If $\mathcal{C}'$ is a quasivariety of σ'-algebras, then ${}^\wedge\mathcal{C}'$ is also a quasivariety of σ-algebras. ([8], p. 274).

We note that there are known infinite many varieties of σ'-algebras $\mathcal{C}'$ such that ${}^\wedge\mathcal{C}'$ is a proper quasivariety. This suggests to look for a description of the set of varieties $\mathcal{C}'$ of σ'-algebras such that ${}^\wedge\mathcal{C}'$ to be also a variety of σ-algebras.

$\underline{4}^o$. Let $\mathcal{C}'$ be a variety of σ'-algebras and $\underline{A}$ be an σ-algebra. Let $\underline{F}'$ be the σ'-algebra which is freely generated by A in $\mathcal{C}'$ and let ρ be the least congruence on $\underline{F}'$ such that:

$$a = f_{\underline{A}}(a_1,\dots,a_n) \text{ in } A \Longrightarrow a \;\rho\; f^\wedge_{\underline{F}'}(a_1,\dots,a_n).$$

Then $A \in {}^\wedge\mathcal{C}'$ if the following condition is satisfied:

$$a,b \in A \Rightarrow (a\rho b \Rightarrow a=b).$$

$\underline{5}^o$. Let $\mathcal{C}' = \mathrm{Var}_{\sigma'}\Sigma'$ be a variety of σ'-algebras defined by a set of identities Σ'. Denote by $\langle\Sigma'\rangle$ the set of identities which are consequences from Σ', i.e. which hold in all σ'-algebras belonging to $\mathcal{C}'$, and denote by ${}^\wedge\Sigma'$ the set of σ-identities $p \equiv q$ such that $p^\wedge \equiv q^\wedge \in \langle\Sigma'\rangle$. Then ${}^\wedge\mathcal{C}'$ is a variety iff ${}^\wedge\mathcal{C}' = \mathrm{Var}\,{}^\wedge\Sigma'$. And, if ${}^\wedge\mathcal{C}'$ is the variety of all σ-algebras then ${}^\wedge\Sigma'$ consists of trivial identities, i.e. the identities of the form $p \equiv p$, where $p \in \sigma_X$.

$\underline{6}^o$ Let $\mathcal{C} = \mathrm{Var}_\sigma\Sigma$, $\mathcal{C}' = \mathrm{Var}_{\sigma'}\Sigma'$ be such that $\mathcal{C} \subseteq {}^\wedge\mathcal{C}'$. Denote by Σ'' the following set of σ'-identities:

$$\{p^\wedge \equiv q^\wedge \mid p \equiv q \in \Sigma\} \cup \Sigma',$$

and let $\mathcal{C}'' = \mathrm{Var}_{\sigma'}\Sigma''$. Then the pair $(\mathcal{C}, \mathcal{C}')$ is $\wedge$-compatible iff $\mathcal{C}' \subseteq \mathcal{C}''$.

$\underline{7}^o$. If $\mathcal{C}'$ is an axiomatizable class of σ'-algebras, then ${}^\wedge\mathcal{C}'$ can be defined by a system of open formulas. ([7]).

$\underline{8}^o$. Let Σ' be a class of σ'-identities satisfyng the following condition:

(**) If u', v' are finite sequences on $\sigma' \cup X$, $p' \in \sigma_X$ and if there is a $q' \in \sigma_X$ such that $u'p'v' \equiv q' \in \langle\Sigma'\rangle$, then there is a $q'' \in \langle\Sigma'\rangle$ such that $u'xv' \equiv q'' \in \langle\Sigma'\rangle$, where x is a variable which does not occur in $u'p'v'$.

Then ${}^\wedge\mathrm{Var}_{\sigma'}\Sigma'$ is a variety of σ-algebras ([5]).

$\underline{2}$. Now, we will state some results concerning special classes of algebras, which will throw better look on the properties $\underline{1}^o$–$\underline{8}^o$.

1) Let $\underline{\mathrm{Sem}}$ be the variety of semigroups. If $\sigma' = \{\cdot\} = \sigma'(2)$ and if $p(x_1,\dots,x_n) \in \sigma_X$, then by the associative law an (2)

identity of the form $p \equiv x_{i_1}x_{i_2}\ldots x_{i_k}$ holds in $\underline{Sem}$, where $i_\nu \in \{1,2,\ldots,n\}$. Thus, we can assume that if $\mathcal{C}$ is a variety of semigroups, then $\mathcal{C} = \mathrm{Var}\Sigma$, where Σ is a set of identities of the forms $x_{i_1}\ldots x_{i_k} \equiv x_{j_1}\ldots x_{j_s}$, where i_ν, $j_\lambda \in \{1,2,\ldots\}$, including the identity $x_1(x_2x_3) = (x_1x_2)x_3$.

The following result is known as Cohn-Rebane's theorem ([1] page 185):

If $\underline{A}$ is an $\mho$-algebra, then there is a semigroup $\underline{S}$ and a mapping $f \mapsto \bar{f}$ of $\mho$ into S such that $A \subseteq S$ and $f_{\underline{A}}(a_1,\ldots,a_n) = \bar{f}a_1\ldots a_n$ for each $f \in \mho(n)$ and all $a_1,\ldots,a_n \in A$. Then we say that $\underline{A}$ is an $\mho$-subalgebra of the semigroup $\underline{S}$. If $\mathcal{C}'$ is a class of semigroups, then by $\mathcal{C}'(\mho)$ will be denoted the class of $\mho$-algebras which are $\mho$-subalgebras of semigroups belonging to $\mathcal{C}'$. Thus, the Cohn-Rebane's theorem can be formulated as follows:

1.1) $\underline{Sem}(\mho)$ is the variety of all $\mho$-algebras.

We will state some other results. First, we will give some definitions. If $p \in \mho_X$ and if $b \in X \cup \mho$, then $|p|_b$ is the number of occurences of the symbol b in p. Also, by $\underline{Absem}$ we denote the variety of commutative semigroups, and by $\underline{C}_{r,m}$ the variety $\underline{Absem}(x^r = x^{r+m})$, where r and m are positive integers. Then we have:

1.2) $\underline{A} \in \underline{Absem}(\mho)$ if $\underline{A}$ satisfies any identity $p \equiv q$, where $p,q \in \mho_X$ are such that $|p|_b = |q|_b$, for each $b \in \mho \cup X$ ([10]).

1.3) $\underline{C}_{r,m}(\mho)$ is a variety iff r=1 or $\mho = \mho(1)$. ([6]).

We note that, if $\mho(o) = \emptyset$, then 1.1) and 1.2) are consequences from $\underline{8}^o$. If in 1.1) or 1.2) we have $\mho(o) = \emptyset, \mho \setminus \mho(o) \neq \emptyset$ (or in 1.3) $\mho \neq \emptyset$), then the condition (**) of $\underline{8}^o$ is not satisfied.

2) Let $\mho = \{f\} = \mho(n)$, $\mho' = \{\cdot\} = \mho'(2)$ and $f^\wedge = x_1x_2\ldots x_n$. If $\mathcal{C}'$ is a class of groupoids, then $^\wedge\mathcal{C}'$ is denoted by $\mathcal{C}'(n)$. Also, $\underline{Sem}$ (xyz=xyxz, xyz=xzyz), $\underline{Sem}$ (xyz=xyxz), $\underline{Sem}(x^r = x^{r+m})$ will be denoted respectively by: $\underline{D}$, $\underline{D}^\ell$, $\underline{P}_{r,m}$. And, $\underline{Sem}_n$ is the class of n-semigroups, i.e. algebras with an associative n-ary operation.

2.1) $\underline{Sem}(n) = \underline{Sem}_n$.

2.2) $\underline{P}_{r,m}(n)$ is a variety iff r=1 or n-1 is a divisor of m.

2.3) $\underline{C}_{r,m}(n)$ is a variety for all r,m,n.

2.4) $\underline{D}(n)$ is a variety for every n.

2.5) $\underline{D}^\ell(n)$ is a proper quasivariety for every $n \geq 3$.

2.6) Let Σ' be a set of semigroup identities $p \equiv q$ such that

$$|p|_i \equiv |q|_i \pmod{n-1} \qquad (***)$$

for each i=1,2,... , where $n \geq 3$, and let $\mathcal{C}' = \underline{Sem}(\Sigma)$. Then $\mathcal{C}'(n)$ is a variety. (We note that this result is a corrolary from $\underline{8}^o$; and, conversely, if a variety $\mathcal{C}' = \underline{Sem}(\Sigma')$ satisfies the condition (**) of $\underline{8}^o$, then (***) is satisfied for every identity $p \equiv q \in \Sigma'$.)

The above results are proved in the papers [3], [4], [5], [9]. Some of the results in 1) and 2) suggest the following conjecture: If $\mathcal{C}'$ is a variety of semigroups such that $\mathcal{C}(\mathcal{O})$ is a variety of $\mathcal{O}$-algebras for every $\mathcal{O}$, then $\mathcal{C}'(n)$ is a variety of n-semigroups for every $n \geq 2$.

3) If $\underline{R}$ is a ring, then by 1.1) there is a semigroup $\underline{S}$ and a pair of elements $a,b \in S$ such that $x+y = axy$, $x\bullet y = bxy$ ("$\bullet$" is the multiplication in the ring $\underline{R}$). But, if $\underline{S}$ is a semigroup with at least two elements, and if the operations + and $\bullet$ defined on $\underline{S}$ by: $x+y = axy$, $x\bullet y = bxy$, where $a,b \in S$, then $(S;+,\bullet)$ is never a ring. This example shows that it can happen a pair $(\mathcal{C}, \mathcal{C}')$ to be not $\wedge$-compatible, although $\mathcal{C} \subseteq {}^{\wedge}\mathcal{C}'$. In [2] there are given several examples of such noncompatible pairs. We note that in each of the examples 1.1)-1.3), 2.1)-2.4) we have a compatible pair of varieties.

4) Now we will finish our considerations by an example of a variety $\mathcal{C}' = \mathrm{Var}\Sigma'$ such that ${}^{\wedge}\mathcal{C}'$ is not a variety although ${}^{\wedge}\Sigma'$ does not contain non trivial identities. Namely, let $\mathcal{O}' = \mathcal{O}'(2) = \{\cdot\}$, $\mathcal{O} = \mathcal{O}(3) = \{f\}$, and $f^{\wedge} = (x_1x_2)x_3$. If $\Sigma' = \{(((x_1x_2)x_1)x_2)x_1 = ((x_1x_1)x_1)(x_2x_2)\}$, then ${}^{\wedge}\Sigma'$ does not contain nontrivial identities, but ${}^{\wedge}\mathrm{Var}\Sigma'$ is a proper subclass of the class of ternary groupoids (i.e. algebras with a ternary operation).

R E F E R E N C E S

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