SUBALGEBRAS OF ABELIAN TORSION GROUPS

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G.Čupona, G.Vojvodić and S.Crvenković

An Ω-algebra $\underline{A}=(A;\Omega)$ is said to be an Ω-subalgebra of a semigroup S if $A\subset S$ and there is a mapping $\omega\mapsto\bar\omega$ of Ω into S such that $\omega(a_1,\dots,a_n)=\bar\omega a_1\dots a_n$ for each n-ary operator $\omega\in\Omega$ and any $a_1,\dots,a_n\in A$. If $\underline{C}$ is a class of semigroups, then by $\underline{C}(\Omega)$ is denoted the class of Ω-algebras which are Ω-subalgebras of semigroups belonging to $\underline{C}$. Here we give corresponding descriptions of the classes ABTG(Ω) and $A_m(\Omega)$, where ABTG is the class of abelian torsion groups and A_m the class of abelian groups in which each element has an order which is a divisor of m ($m\geq 2$ is a given integer).

1. First, we will give a description of ABTG(Ω).

Theorem 1. *Let* $\Omega\neq\Omega(1)$ ($\Omega(n)$ *is the set of* n-*ary operators belonging to* Ω). *An* Ω-*algebra* $\underline{A}=(A;\Omega)$ *belongs to* ABTG(Ω) *iff it satisfies the following conditions:*

(*) *For every* $m,n\geq 1,\omega'\in\Omega(m),\omega''\in\Omega(n),i\in N_m=\{1,2,\dots m\}$ *and permutation* $\nu\mapsto i_\nu$ *of* N_m *the following identity equations are satisfied:*

$$\omega'(x_1,\dots,x_m)=\omega'(x_{i_1},\dots,x_{i_m}),$$

$$\omega'\omega''(x_1,\dots,x_{m+n-1})=\omega''\omega'(x_1,\dots,x_{m+n-1})=$$
$$=\omega'(x_1,\dots,x_{i-1},\omega''(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{m+n-1});$$

(**) *There is a mapping* $m:z\mapsto m(z)$ *of* $A\cup\Omega$ *into the set of positive integers such that:*

$$\omega_1^{i_1}\dots\omega_p^{i_p}(a_1^{\alpha_1},\dots,a_q^{\alpha_q})=\omega_1^{j_1}\dots\omega_p^{j_p}(a_1^{\beta_1},\dots,a_q^{\beta_q}),$$

for any $\omega_\nu\in\Omega(n_\nu),a_\lambda\in A$ *and nonnegative integers* $i_\nu,j_\nu,\alpha_\lambda,\beta_\lambda$ *such that:*

$$i_\nu\equiv j_\nu(\mathrm{mod}\, m(\omega_\nu)),\ \ \alpha_\lambda\equiv\beta_\lambda(\mathrm{mod}\, m(a_\lambda))$$

and:

$$1+i_1n_1+\dots+i_pn_p=\alpha_1+\dots+\alpha_q$$
$$1+j_1n_1+\dots+j_pn_p=\beta_1+\dots+\beta_q.$$

Proof. In the first place, it is clear that if $\underline{A}$ is a subalgebra of an abelian semigroup S then (*) is satisfied. (In [4] it is shown that the converse is also satisfied). If $S \in ABTG$ and if for each $a \in A (\omega \in \Omega)$, $m(a) (m(\omega))$ is the order of a $(\bar{\omega})$ in S, we obtain that the condition (**) is satisfied.

Assume now that $\underline{A}=(A;\Omega)$ is an Ω-algebra which satisfies the conditions (*) and (**). If $z \in A \cup \Omega$ then C_z denotes the cyclic group with a generator z and order $m(z)$. Further on, let H be the free product $\mathsf{H}=\coprod_{z \in A\cup\Omega} C_z$ in the class of abelian groups. (We use a multiplicative notation.)

If $u=au' \in \mathsf{H}$ and $a=\omega(a_1,\dots,a_n)$ in $\underline{A}$, then we write $u \vdash \omega a_1 \dots a_n u'$, and also $\omega a_1 \dots a_n u' \dashv u$. Let $u \vdash\!\dashv v \Leftrightarrow u \vdash v$ or $u \dashv v$. Further on, denote by $\approx$ the reflexsive and transitive extension of $\vdash\!\dashv$, i.e.:

$$u \approx v \Leftrightarrow (\exists u_0,u_1,\dots,u_p \in \mathsf{H})\, u=u_0, v=u_p, p>0 \text{ and } u_{i-1} \vdash\!\dashv u_i$$

for each $i \in \{1,\dots,p\}$.

Then, clearly, $\approx$ is a congruence on H, and $\omega(a_1,\dots,a_n)= =a$ in $\underline{A} \Rightarrow \omega a_1 \dots a_n \approx a$.

We will show that:

(Δ) $\quad a,b \in A \Rightarrow (a \approx b \Rightarrow a=b)$,

and this will complete the proof of Theorem 1..

First we introduce the notion of Ω-word. Namely, an element $w \in \mathsf{H}$ is said to be an Ω-word iff

$$w=\omega_1^{i_1}\omega_2^{i_2}\dots\omega_p^{i_p}a_1^{\alpha_1}\dots a_q^{\alpha_q},$$

and $1+i_1n_1+\dots+i_pn_p=\alpha_1+\dots+\alpha_q$, where $\omega_\nu \in \Omega(n_\nu+1)$, $i_\nu, \alpha_\lambda > 0$.

Then,

$$\omega_1^{i_1}\dots\omega_p^{i_p}(a_1^{\alpha_1},\dots,a_q^{\alpha_q})=a \in A,$$

and we say that $a=[w]$ is the "value" of w.

We note that by (**) the value of an Ω-word w is uniquelly determined.

Clearly, (Δ) is a consequence of the following proposition

$(\Delta\Delta)$ Let $u,v \in \mathsf{H}$ be such that $u \vdash\!\dashv v$. If u is an Ω-word then v is also an Ω-word and $[u]=[v]$.

<u>Proof.</u> Let $u=\omega_1^{i_1}\dots\omega_p^{i_p}a_1^{\alpha_1}\dots a_q^{\alpha_q}$, $\omega_\nu \in \Omega(n_\nu+1)$ and $1+i_1n_1+\dots+$

$+i_p n_p = \alpha_1 + \ldots + \alpha_q$.

Assume first that $u \vdash v$. Then $u = \omega_1^{i_1} \ldots \omega_p^{i_p} a_1^{\beta_1} a_2^{\alpha_2} \ldots a_q^{\alpha_q}$, $\beta_1 \equiv \alpha_1 (\mathrm{mod}\, m(a_1))$, $\beta_1 \geq 1$ \ $a_1 = \omega_1(a_1^{\gamma_1}, \ldots, a_q^{\gamma_q})$, $\gamma_\lambda \geq 0$,

$$v = \omega_1^{i_1+1} \omega_2^{i_2} \ldots \omega_p^{i_p} a_1^{\beta_1+\gamma_1-1} a_2^{\alpha_2+\gamma_2} \ldots a_q^{\alpha_q+\gamma_q}.$$

Let $\omega \in \Omega(n+1)$, $n \geq 1$. Then $v = \omega^{sm(\omega)} a_1^{tm(a_1)} v$, for each s, $t \geq 0$ and it can be easily seen that there exist $s, t \geq 0$ such that $1 + sm(\omega)n + (i_1+1)n_1 + i_2 n_2 + \ldots + i_p n_p = tm(a_1) + \beta_1 + \gamma_1 - 1 + \alpha_2 + \gamma_2 + \ldots + \alpha_q + \gamma_q$, and this will imply that v is also an Ω-word. Moreover, we shall have:

$$[v] = \omega^{sm(\omega)} \omega_1^{i_1+1} \omega_2^{i_2} \ldots \omega_p^{i_p} (a_1^{tm(a_1)+\beta_1+\gamma_1-1}, a_2^{\alpha_2+\gamma_2} \ldots a_q^{\alpha_q+\gamma_q})$$

$$= \omega^{sm(\omega)} \omega_1^{i_1} \ldots \omega_p^{i_p} (a_1^{tm(a_1)+\beta_1}, a_2^{\alpha_2}, \ldots, a_q^{\alpha_q}) =$$

$$= \omega_1^{i_1} \ldots \omega_p^{i_p} (a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots, a_q^{\alpha_q}) = [u].$$

Consider now the case $u \dashv v$. Namely, we can assume that $u = \omega_1^{j_1} \omega_2^{i_2} \ldots \omega_p^{i_p} a_1^{\beta_1} \ldots a_q^{\beta_q}$, $j_1 \geq 1$, $j_1 \equiv i_1 (\mathrm{mod}\, m(\omega_1))$, $\beta_\lambda \equiv \alpha_\lambda (\mathrm{mod}\, m(a_\lambda))$, and $v = \omega_1^{j_1-1} \omega_2^{i_2} \ldots \omega_p^{i_p} a_1^{\beta_1-\gamma_1+1} a_2^{\beta_2-\gamma_2} \ldots a_q^{\beta_q-\gamma_q}$, where $a_1 = \omega_1(a_1^{\gamma_1}, \ldots, a_q^{\gamma_q})$. We can also assume that $n_k \geq 1$ for some $k \in \{1, \ldots, p\}$. Now it can be easily seen that there exist $s_1, s_2, \ldots, s_p, t_1, \ldots, t_q \geq 0$ such that

$$1 + (j_1 - 1 + s_1 m(\omega_1)) n_1 + (i_2 + s_2 m(\omega_2)) n_2 + \ldots + (i_p + s_p m(\omega_p)) n_p =$$

$$= (\beta_1 - \gamma_1 + 1 + t_1 m(a_1)) + (\beta_2 - \gamma_2 + t_2 m(a_2)) + \ldots + (\beta_q - \gamma_q + t_q m(a_q)).$$

Then:

$$v = \omega_1^{j_1-1+s_1 m(\omega_1)} \ldots \omega_p^{i_p+s_p m(\omega_p)} a_1^{\beta_1-\gamma_1+1+t_1 m(a_1)} a_2^{\beta_2-\gamma_2+t_2 m(a_2)} \ldots a_q^{\beta_q-\gamma_q+t_q m(a_q)}$$

and

$$[v] = \omega_1^{j_1-1+s_1 m(\omega_1)} \ldots \omega_p^{i_p+s_p m(\omega_p)} (\omega_1(a_1^{\gamma_1}, \ldots, a_q^{\gamma_q}), a_1^{\beta_1-\gamma_1+t_1 m(a_1)}, \ldots, a_q^{\beta_q-\gamma_q+t_q m(a_q)}) =$$

$$= \omega_1^{j_1+s_1 m(\omega_1)} \ldots \omega_p^{i_p+s_p m(\omega_p)} (a_1^{\beta_1+t_1 m(a_1)}, \ldots, a_q^{\beta_q+t_q m(a_q)}) =$$

$$= \omega_1^{i_1} \ldots \omega_p^{i_p} (a_1^{\alpha_1}, \ldots, a_q^{\alpha_q}) = [u].$$

This completes the proof of $(\Delta\Delta)$, and thus of Theorem1. as well.

Corollary. *If* $\Omega \setminus \Omega(1) \neq \emptyset$, *then the class* $A_m(\Omega)$ *is a variety.*

Proof. In this case we have that $m: z \mapsto m(z) = m$ is a constant, and thus in (**) we have a system of identities.

2. Consider now the case when $\Omega = \Omega(1)$ consists of only unary operators. The following example shows that the conditions (*), (**) are not suficient.

Example. Let $\Omega=\Omega(1)$ and let ω_0 be a fixed element of Ω. Let $A=\{1,2,3,4,5\}$ and the algebra $\underline{A}=(A;\Omega)$ be defined by:

$$\omega_{0_{\underline{A}}}=(123)(45),\ \omega_{\underline{A}}=1_A \quad \text{if} \quad \omega\neq\omega_0.$$

The algebra $\underline{A}$ satisfies the conditions (*),(**). Namely, the condition (*) reduces to the commutativity of the semigroup generated by the transformations which are interpretations of the operators from Ω. And, if we put $m(\omega_0)=6, m(\omega)=m(a)=1$ for each $\omega\in\Omega$, $\omega\neq\omega_0$ and each $a\in A$, we obtain that (**) is also satisfied. But $\underline{A}$ does not belong to ABTG(Ω), for if $\underline{A}$ were an Ω-subalgebra of a group $G\in$ ABTG then we would have $\bar{\omega}_0^2 4=4$, but $\bar{\omega}_0^2 1=3$, which is impossible. (Namely, $\bar{\omega}_0^2 4=4$ implies that $\bar{\omega}_0^2$ is the identity of the group G.)

Theorem 2. *Let $\Omega=\Omega(1)$. An Ω-algebra $\underline{A}=(A;\Omega)$ belongs to ABTG(Ω) iff it satisfies the following conditions:*

(*´) $\omega'\omega''(x)=\omega''\omega'(x)$, *for any* $\omega',\omega''\in\Omega, x\in A$;

(**´) *There is a mapping* $m:\omega\mapsto m(\omega)$ *of* Ω *into the set of positive integers such that* $\omega^{m(\omega)}(x)=x$, *for any* $\omega\in\Omega$, $x\in A$;

(***´) $\underline{A}$ *satisfies any quasiidentity of the following form:*

$$\omega_1\ldots\omega_p(x)=\omega_1'\ldots\omega_q'(x)\Rightarrow\omega_1\ldots\omega_p(y)=\omega_1'\ldots\omega_q'(y),$$

Proof. Clearly, the conditions (*´),(**´) and (***´) are necessary. The suficiency is a corollary of the following

Lemma. Let Γ be a commutative group of permutations on a set A such that:

$$\omega'(x)=\omega''(x)\Rightarrow\omega'(y)=\omega''(y).$$

Define a relation $\approx$ on A by:

$a\approx b \Leftrightarrow (\exists\phi\in\Gamma)b=\phi(a)$. Then, (i) $\approx$ is an equivalence in A.

(ii) If B is a subset of A such that $(\forall a\in A)(\exists! b\in B)a\approx b$ then the mapping $\xi:(\omega,b)\mapsto\omega(b)$ is a bijection from $\Omega\times B$ into A, such that $\xi(\omega'\omega'',b)=\omega'(\xi(\omega'',b))$.

(iii) If K is an abelian group generated by B and if $G=\Gamma\times K$, then by putting $\xi(\omega,b)=(\omega,b)$, we obtain that the algebra $(A;\Gamma)$ is a Γ-subalgebra of G.

The proof of the Lemma is obvious.

3. Here we will make some remarks and state some problems.

First, we note that if we try to generalize Theorem 1. for abelian periodical semigroups, then we get the result that this generalization is not true. And, we do not know if the corresponding analogy of Theorem 1. holds for the class of commutative semigroups with the propertiy $(\forall x)(\exists m>0)x^{m+1}=x$.

The similar situation arises if we try to generalize Theorem 2..

We also note that we do not know any convenient description of the class $\underline{C}(\Omega)$ if $\underline{C}$ is one of the following classes of semigroups:

(a) idempotent semigroups; (b) periodic groups;
(c) groups; (d) inverse semigroups;
(e) regular semigroups; (f) completely simple semigroups.

In other words we think that the well known Kurosh's problem of characterisations of $\underline{C}(\Omega)$ is until now solved only for a few classes of semigroups, namely only if $\underline{C}$ is one of the following classes of semigroups:

1) semigroups [1]; 2) commutative semigroups [4];
3) cancelative semigroups [5]; 4) nilpotent semigroups [6];
5) semilattices [2];
6) $\underline{C}_{1,m}$, i.e. the class of commutative semigroups that satisfies the identity $x^{m+1}=x$ [3];
7) ABTG; 8) A_m.

We would like also to mention the problem of finding the set of varieties $\underline{C}$ of semigroups such that $\underline{C}(\Omega)$ is also a variety for all Ω or for some Ω.

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