THE PROBLEM OF SOLVABILITY OF POLYLINEAR REPRESENTATIONS OF UNIVERSAL ALGEBRAS IN SEMIGROUPS

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The problem of effectivness of different kinds of embeddings of universal algebras in semigroups is treated in this paper.

1. Consider an Ω-algebra $\mathbf{A} = (A; \Omega)$, i.e. $\Omega = \cup\{\Omega(n) \mid n \geqslant 1\}$ is a set of finitary operators such that $n \neq m \Rightarrow \Omega(n) \cap \Omega(m) = \varnothing$, and every n-ary operator $\omega \in \Omega(n)$ induces an n-ary operation $\omega_{\mathbf{A}}$ on A. (We will use the same notation for an operator and the corresponding operation in the algebra, i.e. we will write $\omega(a_1, \ldots, a_n)$ instead of $\omega_{\mathbf{A}}(a_1, \ldots, a_n)$.) We associate three semigroups to the algebra $\mathbf{A}$ as follows:

$$\mathbf{A}_1^{\triangle} = < A \cup \Omega \mid \{ a = \omega a_1 \ldots a_n \mid a = \omega(a_1, \ldots, a_n) \text{ in } \mathbf{A}\} > \quad (1.1)$$

$$\mathbf{A}_2^{\triangle} = < A \cup \Omega^{\wedge} \mid \{a = \omega_0 a_1 \omega_1 \ldots a_n \omega_n \mid a = \omega(a_1, \ldots, a_n) \text{ in } \mathbf{A}\} > \quad (1.2)$$

$$\mathbf{A}_3^{\triangle} = < A \cup \Omega \mid \{a = a_1 \omega a_2 \ldots a_n \mid a = \omega(a_1, \ldots, a_n) \text{ in } \mathbf{A}\} > \quad (1.3)$$

It is assumed in (1.2) that for any $\omega \in \Omega(n)$, $\omega^{\wedge} = \{\omega_0, \ldots, \omega_n\}$ is a set with $n+1$ elements such that $\omega^{\wedge} \cap \tau^{\wedge} \neq \varnothing \Rightarrow \omega = \tau$, and $\Omega^{\wedge} = \cup\{\omega^{\wedge} \mid \omega \in \Omega\}$. It is also assumed that $\Omega(1) = \varnothing$ in (1.3).

We notice that, in all the above presentations, the right-hand sides of the defining relations have greater lengths than the ones on the left-hand sides. So, we can define reduced words to be those words which have no sub-words which are the right-hand sides of the definig relations. It is clear that for any word u there is a reduced word $\bar{u}$, such that $\bar{u}$ is obtained from u by a finite application of the defining relations.

It is easy to prove the following

Theorem 1.1. The irreducible representative $\bar{u}$ for any word u is uniquely defined in (1.1) and (1.2) for every algebra $\mathbf{A}$. Every word has a unique irreducible representative in (1.3) iff the algebra $\mathbf{A}$ statisfies the identities

$$\omega\tau(x_1, x_2, \ldots, x_{m+n-1}) = \tau(x_1, \ldots, x_{m-1}, \omega(x_m, \ldots, x_{m+n-1})) \quad (1.4)$$

where $\omega \in \Omega(n)$, $\tau \in \Omega(m)$. ■

An Ω-algebra $\mathbf{A}$ is said to be recursive iff A and Ω are recursive sets, and every operation $\omega_{\mathbf{A}} : A^n \to A$ induced by $\omega \in \Omega(n)$ is recursive. As a consequence of Theorem 1.1 we have:

Corollary 1.2. If the algebra $\mathbf{A}$ is recursive, then the semigroups $\mathbf{A}_1^{\triangle}$ and $\mathbf{A}_2^{\triangle}$ are also recursive; furthermore, if the algebra $\mathbf{A}$ satisfies the identities (1.4), then $\mathbf{A}_3^{\triangle}$ is recursive as well. ■

Since the elements of the set A are reduced in all of the presentations (1.1), (1.2) and (1.3), we have:

Corollary 1.3. For any Ω-algebra $\mathbf{A}$ there exists a semigroup $\mathbf{S}$ such that $A \cup \Omega \subseteq S$ and the equality

$$\omega(a_1, a_2, \ldots, a_n) = \omega a_1 a_2 \ldots a_n$$

holds for every $\omega \in \Omega(n)$, $a_1, \ldots, a_n \in A$. ■

Corollary 1.4. For any Ω-algebra $\mathbf{A}$ there exists a semigroup $\mathbf{S}$ and a mapping $\omega \to \omega^{\wedge} = (\omega_0, \ldots \omega_n)$ of Ω into $\bigcup_{n=1}^{\infty} S^n$ such that $\omega \in \Omega(n)$

$\Rightarrow \omega^\wedge \in S^{n+1}$, $A \subseteq S$ and the equality

$$\omega(a_1, \ldots, a_n) = \omega_0 a_1 \omega_1 \ldots a_n \omega_n$$

holds for every $\omega \in \Omega(n), a_1, \ldots, a_n \in A$. ■

Corollary 1.5. If the algebra **A** satsfies the identities (1.4), then there exists a semigroup **S** such that $A \cup \Omega \subset S$ and the equality

$$\omega(a_1, a_2, \ldots, a_n) = a_1 \omega a_2 \ldots a_n$$

holds for every $\omega \in \Omega(n)$, $a_1, a_2, \ldots, a_n \in A$. ■

Namely, we can take **S** to be the semigroup $\mathbf{A}_1^{\triangle}$, $\mathbf{A}_2^{\triangle}$ and $\mathbf{A}_3^{\triangle}$ in the corresponding cases.

Remark that Corollary 1.3 is the well known Cohn-Rebane's theorem-([2], [7]) and Corollary 1.5 is proved in [3].

2. We will consider here more general representations of Ω-algebras into semigroups, and (1.1), (1.2) and (1.3) will be special cases of them.

Let Ω be a set of finitary operations, C be a given set and $e \notin \Omega \cup C$. Assume that for any $\omega \in \Omega(n)$ we have a sequence $\omega^{\triangle} = (\omega_0, \omega_1, \ldots, \omega_n)$, where $\omega_i \in C \cup \{e\}$. If **A** is a given Ω-algebra with a carrier A, then we consider the semigroup $\mathbf{A}^{\triangle}$ given by the following presentation:

$$\mathbf{A}^{\triangle} = < A \cup C; \{a = \omega_0 a_1 \omega_1 \ldots \omega_{n-1} a_n \omega_n \mid a = \\ = \omega(a_1, \ldots, a_n) \text{ in } \mathbf{A}\} > \qquad (2.1)$$

We suppose that e is the empty word in (2.1), i.e. if $\omega_i = e$ for some i, then we do not write ω_i on the corresponding right-hand side of the defining relation. (We say that $\triangle$ is the kind of the polylinearity)

It is clear that (1.1), (1.2) and (1.3) are special cases of (2.1). Namely, if $C = \Omega$ and $\omega^{\triangle} = (\omega, e, \ldots, e)$ $(\omega^{\triangle} = (e, \omega, e, \ldots, e))$, we obtain (1.1) ((1.3)). If $\omega_i \neq e$ for every $\omega \in \Omega$ (n), $i \in \{0, 1, \ldots, n\}$ and $\omega_i = \tau_j \Leftrightarrow \omega = \tau, i = j$, then we obtain (1.2).

The reduced words could be defined as above, and so we have

Theorem 2.1. Let the algebra **A** be defined such that for any word u in the presentation (2.1) there exists a unique reduced representative $\bar{u}$.

Then, if the algebra **A** is recursive, the semigroup $\mathbf{A}^{\triangle}$ is recursive as well. ■

We are looking now for conditions under which we can have a unique reduced representatives for a given word.

Define the set of Ω-words, which isa subset of $(A \cup C)^+$ [1], in this inductive way:

(i) every element of A is an Ω-word;

(ii) if $u_1, u_2, \ldots, u_n$ are Ω-words and $\omega \in \Omega(n)$, then $\omega_0 u_1 \omega_1 u_2 \ldots \ldots \omega_{n-1} u_n \omega_n$ is an Ω-word;

(iii) a word $u \in (A \cup C)^+$ is an Ω-word iff it is obtained by a finite application of (i) and (ii).

Let **A** be an Ω-algebra. For every Ω-word u let us define its value $[u] \in A$ as follows:

$a \in A \Rightarrow [a] = a$;

if $\omega \in \Omega(n)$ and $u_1, u_2, \ldots, u_n$ are Ω-words with values

[1]) $\mathbf{B}^+$ is the free semigroup on **B**.

$[u_i] = b_i$, $i = 1, 2, \dots, n$, then $b = \omega(b_1, b_2, \dots, b_n)$ is one value of the Ω-word $u = \omega_0 u_1 \omega_1 \dots \omega_{n-1} u_n \omega_n$. Thus the value of an Ω-word need not be uniquely defined.

It is clear what we mean by „an Ω-word u is a maximal Ω-subword of a given word v". (Note that u can have both maximal and non maximal appearences in v.)

We can formulate now the wanted condition:

Theorem 2.2. Let the algebra **A** and $\triangle$ satisfy the conditions:

1) Every word $v \in (A \cup C)^+$ can be represented uniquely in the form

$$v = \alpha_0 u_1 \alpha_1 u_2 \dots \alpha_{p-1} u_p \ \alpha_p \tag{2.2}$$

where $\alpha_v \in C^*$[1] and $u_1, u_2, \dots, u_p$ are maximal Ω-subwords of v. (We say that (2.2) is a canonical representation of v.)

2) Every Ω-word u has a uniquelly defined value $[u]$.

Define a relation $\approx$ on $(A \cup C)^+$ as follows: $v \approx w$ iff v has a canonical representation of the form (2.2), w has a canonical representation

$$w = \alpha_0 \ u'_1 \ \alpha_1 u'_2 \dots \alpha_{p-1} \ u'_p \alpha_p$$

and $[u'_i] = [u_i]$ for $i = 1, 2, \dots, p$.

Then $\approx$ is an equivalence on $(A \cup C)^+$.

If it is satisfied the condition

3) $\approx$ is a congruence on the semigroup $(A \cup C)^+$,

then $\mathbf{A}^\triangle$ is isomorphic to $(A \cup C)^+ / \approx$ and every word v with a canonical representation (2.2) has uniquelly defined reduced representation

$$\bar{v} = \alpha_0 a_1 \alpha_1 a_2 \dots \alpha_{p-1} a_p \ \alpha_p \tag{2.2}$$

where $[u_i] = a_i$, $i = 1, 2, \dots, p$. ■

The condition 3) is independent from 1) and 2). Namely, let $\Omega = \{\tau, \omega\}$, where $\omega \in \Omega(3)$, $\tau \in \Omega(4)$, and let $\omega^\triangle = (e, e, e, e)$, $\tau^\triangle = (e, e, e, e, e,)$, $C = \emptyset$. Then if $u \in A$ or $u = a_1 a_2 \dots a_p$, $p \geqslant 3$, u is an Ω-word, and if $u = ab$, $a, b \in A$, then a and b are maximal subwords of u. Thus, 1) is satisfied for any algebra $\mathbf{A} = (A; \omega, \tau)$. The condition 2) is satisfied iff **A** satisfies the general associative law, i.e. iff **A** is an associative ([1], [4]). But, there are associatives which do not satisfy 3).

For example, let $A = \{a, b, c\}$ and $\omega(x_1, x_2, x_3) = a$ for every $x_1, x_2, x_3 \in A$, $\tau(x_1, x_2, x_3, x_4) = a$ for every $x_1, x_2, x_3, x_4 \in A$ such that $x_i \neq c$ for at least one i, and $\tau(c, c\ c, c) = b$. Then $(A; \omega, \tau)$ is an associative ([4]), but the relation $\approx$ is not a congruence, since $c^3 \approx a^3$, $c^4 \approx b$, $a^3 c \approx a$, but a and b are not equivalent.

Notice that the condition 1) depends on $\triangle$, but not on the considered algebra, and that 2) is satisfied iff the algebra **A** satisfies some corresponding system of identities. Thus 2) is satisfied in (1.3) iff the algebra **A** satisfies the identities (1.4).

3. Assume that the Ω-algebra **A** has a presentation

$$\mathbf{A} = < B; \Lambda > \tag{3.1}$$

in the class of all Ω-algebras, or in some variety of Ω-algebras. We want to give a presentation of the semigroup $\mathbf{A}^\triangle$, for a given $\triangle$.

Define the set Ω_A of Ω-terms without variables to be the intersection of all subsets H of $(A \cup C)^+$ with the properties

(i) $A \subseteq H$

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1) $\mathbf{B}^*$ is the free monoid on B.

(ii) $\omega \in \Omega(n), \quad \xi_1, \ldots, \xi_n \in H \Rightarrow \omega \xi_1 \ldots \xi_n \in H$.

For every Ω-term ξ without variables we have an Ω-word $\xi^{\triangle}$, the „translation" of ξ, obtained in this inductive way:

$$a^{\triangle} = a, \text{ for every } a \in A,$$

$$(\omega\, \xi_1 \ldots \xi_n)^{\triangle} = \omega_0\, \xi_1^{\triangle}\, \omega_{1\,1} \ldots \xi^{\triangle}\, \omega_n \quad \text{for every}$$

$$\omega \in \Omega(n), \quad \xi_1, \xi_2, \ldots, \xi_n \in \Omega_A.$$

The translation $\Lambda^{\triangle}$ of Λ is defined by

$$\Lambda^{\triangle} = \{\xi^{\triangle} = \eta^{\triangle} \mid \xi = \eta \in \Lambda\}$$

Now, we have the following results:

Theorem 3.1 If the algebra **A** has the presentation (3.1), then the semigroup $\mathbf{A}^{\triangle}$ has the presentation

$$\mathbf{A}^{\triangle} = < B \cup C \mid \Lambda^{\triangle} > \blacksquare \tag{3.2}$$

Theorem 3.2. Suppose that the conditions of Theorem 2.2 are satisfied. Then the presentation (3.1) is solvable iff the presentation (3.2) is solvable. ■

We notice that an *n*-group is recursive iff its universal covering is recursive ([6], [5]), but it is not known whether the same result is true when *n*-semigroups are considered.

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ПРОБЛЕМОТ НА РЕШЛИВОСТ НА ПОЛИЛИНЕАРНИТЕ ПРЕТСТАВУВАЊА НА УНИВЕРЗАЛНИ АЛГЕБРИ ВО ПОЛУГРУПИ

Една Ω-алгебра (A, Ω) е полилинеарна подалгебра од полугрупа $(S, \,.\,)$ акко $A \subseteq S$ и за секои $\omega \in \Omega$, $a_1, a_2, \ldots, a_n \in A$,

$$(*) \qquad \omega(a_1, a_2, \ldots, a_n) = \omega_0\, a_1\, \omega_1\, a_2\, \omega_2 \ldots a_n\, \omega_n,$$

каде на секое $\omega \in \Omega(n)$ му одговара низа $(\omega_0, \ldots, \omega_n)$ од елементи од S, при што се дозволува некои ω_i да не се јавуваат во (*). Ро работава се разгледуваат повеќе видови полилинеарни сместувања на универзални алгебри во полугрупи, при што главен акцент е ставен на ефикасноста на сместувањето. Основен резултат на работава е Теорема 2.2.