

COHN-REBANE THEOREM FOR VECTOR VALUED ALGEBRAS

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Dedicated to Academician Petar Serafimov on occasion of his 70-th anniversary

A vector valued variant of the well-known Cohn-Rebane's Theorem is given in this paper.

0. Necessary preliminaries will be given first, and then the main result of the paper will be stated.

Let A be a non empty set, n, m positive integers, and f a mapping from A^n into A^m , where A^s is the s -th cartesian power* of A . Then we say that f is an (n, m) -operation on A , and write $\delta(f) = n, \rho(f) = m$; f will also be called a vector valued operation on A . If F is a set of vector valued operations on A , then $(A; F)$ is called a vector valued algebra or, shortly, a v.v.a.

A v.v.a. $(Q; f)$ with an $(m+k, m)$ -operation f , where $m, k \geq 1$, is said to be an $(m+k, m)$ -semigroup if the following equation

$$f(f(x_1^{m+k} x_{m+k+1}^{m+2k})) = f(x_1^j f(x_{j+1}^{j+m+k} x_{j+m+k+1}^{m+2k})) \quad (0.1)$$

is an identity on Q , for every $j \in N_k$, where $N_r = \{1, 2, \dots, r\}$.

Thus: $(Q; f)$ is a $(2,1)$ -semigroup iff it is a usual (binary) semigroup.

The general associative law holds in an $(m+k, m)$ -semigroup. Namely, if $(Q; f)$ is an $(m+k, m)$ -semigroup, and s is a positive integer, an $(m+sk, m)$ -operation f^s can be defined by $f^1 = f$, and

$$f^{s+1}(x_1^{m+(s+1)k}) = f^s(f(x_1^{m+k} x_{m+k+1}^{m+(s+1)k})). \quad (0.2)$$

Then:

0.1. The following equation

$$f^s(x_1^j f^t(y_1^{m+tk} x_{j+1}^{sk})) = f^{s+t}(x_1^j y_1^{m+tk} x_{j+1}^{sk}) \quad (0.3)$$

is an identity on Q , for any $s, t \geq 1$ and $j \in N_{sk}$.

0.2. $(Q; f^s)$ is an $(m+sk, m)$ -semigroup for every $s \geq 1$.

Further, if $(Q; f)$ is an $(m+k, m)$ -semigroup, we will write $[x_1^{m+sk}]$ instead of $f^s(x_1^{m+sk})$, and, sometimes, $[x_1^m]$ instead of (x_1^m) .

Now we can state the main result of this paper.

Theorem. Let $(A; F)$ be a v.v.a. and let m be a positive integer such that $\delta(f) \geq m, \rho(f) \geq m$, for every $f \in F$. There is an $(m+1, m)$ -semigroup $(Q; [\])$ and a mapping $\alpha: f \mapsto \mathbf{f}$ from F into Q such that $A \subseteq Q$ and

$$f(a_1^{\delta(f)}) = (b_1^{\rho(f)}) \Leftrightarrow [\mathbf{f} a_1^{\delta(f)}] = [b_1^{\rho(f)}] \quad (0.4)$$

for any $a_\nu, b_\lambda \in A, f \in F$.

We will prove the Theorem in the next two parts. From now on, we assume that $(A; F)$ is a given v.v.a., and m a positive integer such that $\delta(f) \geq m, \rho(f) \geq m$ for every $f \in F$. Also, $\alpha: f \mapsto \mathbf{f}$ will be a bijection from F onto $\mathbf{F} = \{\mathbf{f} \mid f \in F\}$, $A \cap \mathbf{F} = \emptyset$ and $B = A \cup \mathbf{F}$.

1. Here we will consider the case $m = 1$.

Let $B^+ (B^*)$ be the free semigroup (monoid) on B , i.e.

$$B^+ = \{b_1 \dots b_t \mid b_\nu \in B, t \geq 1\}, \quad B^* = B^+ \cup \{1\}$$

where 1 is the empty sequence on B .

Define a mapping dg from B^* into the set of positive integers in the following way:

$$dg(1) = 0; \quad a \in A \Rightarrow dg(a) = 0; \quad f \in F \Rightarrow dg(\mathbf{f}) = 1 \\ u, v \in B^* \Rightarrow dg(uv) = dg(u) + dg(v).$$

An element $u \in B^+$ is called reducible iff u has a form $u =$

* The elements of A^s will be denote by (α^s) , and here we use the abbreviation x_α^β for $x_\alpha x_{\alpha+1} \dots x_\beta$ if $\alpha \leq \beta$, and x_α^β will be "empty" if $\alpha > \beta$.

$u'f a_1 \dots a_{\delta(f)} u''$, where $u', u'' \in B^*$, $f \in F$, $a_\nu \in A$. And, if $u \in B^+$ is not reducible, then we say that u is a reduced element. Denote the set of all reduced elements of B^+ by B^\wedge . We will define a reduction mapping $\varphi : B^+ \rightarrow B^\wedge$ as follows.

First, if $u \in B^\wedge$ then $\varphi(u) = u$.

Assume now that $u \in B^+$ is a reducible element, and that for every $v \in B^+$ such that $dg(v) < dg(u)$ the reduction $\varphi(v) \in B^\wedge$ is well determined, and furthermore:

$$v \neq \varphi(v) \Leftrightarrow v \in B^+ \setminus B^\wedge \Leftrightarrow dg(\varphi(v)) < dg(v).$$

From the reducibility of u it follows that there exist uniquely determined elements $u', u'' \in B^*$, $f \in F$, $a_1, \dots, a_{\delta(f)} \in A$ such that $u = u' f a_1 \dots a_{\delta(f)} u''$ and ($u' = 1$ or $u' \in B^\wedge$). If $w = u' b_1 \dots b_{\delta(f)} u''$, where $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$, then $dg(w) = dg(u) - 1$, and thus $\varphi(w)$ is well defined element of B^\wedge . Now we define $\varphi(u)$ by

$$\varphi(u) = \varphi(w)$$

Thus $\varphi : B^+ \rightarrow B^\wedge$ is well determined and, by induction, it can easily be seen that the following statements are satisfied:

1.1. $(\forall u \in B^+) [u \notin B^\wedge \Leftrightarrow \varphi(u) \neq u \Leftrightarrow dg(\varphi(u)) < dg(u)]$.

1.2. $(\forall u, v \in B^+) \varphi(uv) = \varphi(\varphi(u) v) = \varphi(u\varphi(v)) = \varphi(\varphi(u) \varphi(v))$.

Define an operation \bullet on B^\wedge in the following way:

$$u, v \in B^\wedge \Rightarrow u \bullet v = \varphi(uv)$$

Then, by 1.2, we find that

1.3. $(B^\wedge; \bullet)$ is a semigroup.

We will show that (0.4) is satisfied.

Assume first that $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$ in $(A; F)$. Then we have:

$$f \bullet a_1 \bullet \dots \bullet a_{\delta(f)} = \varphi(f a_1 \dots a_{\delta(f)}) = b_1 \dots b_{\rho(f)} = \varphi(b_1 \dots b_{\rho(f)}) = b_1 \bullet \dots \bullet b_{\rho(f)},$$

i.e. $[f a_1^{\delta(f)}] = [b_1^{\rho(f)}]$ if we use $[]$ -notation.

Conversely, if $f \in F$, $a_\nu, b_\lambda \in A$ are such that

$$f \bullet a_1 \bullet \dots \bullet a_{\delta(f)} = b_1 \bullet \dots \bullet b_{\rho(f)}$$

then we have

$$\varphi(f a_1 \dots a_{\delta(f)}) = \varphi(b_1 \dots b_{\rho(f)}), \text{ i.e. } f(a_1^{\delta(f)}) = (b_1^{\rho(f)}).$$

This completes the proof of the Theorem in the case $m = 1$.

2. We assume that $m \geq 2$ in this part of the paper.

Define a dimension dm of the elements of a monoid X^* as follows:

$$dm(1) = 0; x \in X \Rightarrow dm(x) = 1; u, v \in X^* \Rightarrow dm(uv) = dm(u) + dm(v).$$

Let $B_0 = B$, $C_p = \{u \in B_p^* \mid dm(u) \geq m+1\}$, $B_{p+1} = B_p \cup C_p \times N_m$ and $\mathbf{B} = \bigcup_{p \geq 0} B_p$.

Thus we have:

$$u \in \mathbf{B} \text{ iff } (u \in B \text{ or } u = (v, i), v \in B^+, dm(v) \geq m+1, i \in N_m).$$

In what follows we will use the following notations:

(i) a, b, c (with or without indices) $\in A$.

(ii) f, g, h (with or without indices) $\in F$.

(iii) x, y, z, u, v, w (with or without indices) $\in B^*$.

(iv) (x, i) will always mean that $i \in N_m$, and $x \in B^*$ is such that $dm(x) > m$, i.e. $(x, i) \in \mathbf{B}$.

Define a degree dg and a length $||$ of the elements of \mathbf{B}^* as follows:

$$dg(a) = 0; dg(f) = 1; dg(uv) = dg(u) + dg(v); dg(u, i) = dg(u).$$

$$|a| = |f| = 1; |uv| = |u| + |v|; |(u, i)| = |u|.$$

An element $u \in \mathbf{B}$ is said to be reducible iff

1) there is an appearance of $f a_1 \dots a_{\delta(f)}$ in u , for some $f, a_1, \dots, a_{\delta(f)}$, or

2) there is an appearance of $(y, 1) (y, 2) \dots (y, m)$ in u , for some y .

And, if $u \in \mathbf{B}$ is not reducible, then it is called reduced. The set of reduced elements of \mathbf{B} will be denoted by B^\wedge : We note that $B \subseteq B^\wedge$.

Define a reduction mapping $\varphi : \mathbf{B} \rightarrow B^\wedge$ as follows:

First, if $u \in B^\wedge$, then we put $\varphi(u) = u$.

Let $u = (x, i) \in \mathbf{B} \setminus B^\wedge$ and assume that for every $v \in \mathbf{B}$ which satisfies the condition

$$dg(v) < dg(u) \quad \text{or} \quad (dg(v) = dg(u) \ \& \ |v| < |u|) \quad (2.1)$$

$\varphi(v)$ is a well determined element of B^\wedge , and that the following statement holds:

$$\begin{aligned} & \varphi(v) \neq v \Leftrightarrow v \in \mathbf{B} \setminus B^\wedge \Leftrightarrow \\ & \Leftrightarrow [dg(\varphi(v)) < dg(v) \quad \text{or} \quad (dg(\varphi(v)) = dg(v) \ \& \ |\varphi(v)| < |v|)] \quad (2.2) \end{aligned}$$

Let $x = x_1 x_2 \dots x_r$, $x_v \in \mathbf{B}$. Then $\varphi(x_j)$ is well defined for any $j \in N_r$. Denote $\varphi(x_1) \varphi(x_2) \dots \varphi(x_r)$ by $\varphi(x)$. If $\varphi(x) \neq x$, and if we put $v = (\varphi(x), i)$, then (2.1) is satisfied, and thus $\varphi(v)$ is defined. Then we put $\varphi(u) = \varphi(v) = \varphi(\varphi(x), i)$.

Assume now that $\varphi(x) = x$, but x has the following form: $x = x' f a_1 \dots a_{\delta(f)} x''$, where x' is chosen in such a way that it has the minimal possible dimension. Let $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$ in $(A; F)$. If $\rho(f) > m$ or $x' x'' \neq 1$ then $v = (x' b_1 \dots b_{\rho(f)} x'', i) \in \mathbf{B}$ and $dg(v) < dg(u)$. Thus we can put $\varphi(u) = \varphi(v)$. If $\rho(f) = m$ and $x' x'' = 1$, then we put $\varphi(u) = \varphi(f a_1 \dots a_{\delta(f)}, i) = b_1$.

The case remains when $\varphi(x) = x$, but x has not the form $x = x' f a_1 \dots a_{\delta(f)} x''$. By assumption $u = (x, i)$ is reducible. The reducibility of $u = (x, i)$ implies that $x = x' (y, 1) (y, 2) \dots (y, m) x''$, and x', y, x'' are uniquely determined if we assume that x' has the corresponding property of minimality. Then the length of $v = (x' y x'', i)$ is less than $|u|$, and $dg(v) \leq dg(u)$. Therefore, $\varphi(v) \in B^\wedge$ is well defined, and in this case we define $\varphi(u)$ by $\varphi(u) = \varphi(v)$.

In such a way all possible cases are exhausted, and thus we have obtained a well defined mapping $\varphi : \mathbf{B} \rightarrow B^\wedge$.

By using an induction on the degrees and the lengths it can be proved that φ admits the following properties:

2.1. $u \in \mathbf{B} \setminus B^\wedge \Leftrightarrow \varphi(u) \neq u \Leftrightarrow$

$$\Leftrightarrow [dg(\varphi(u)) < dg(u) \quad \text{or} \quad dg(\varphi(u)) = dg(u) \ \& \ |\varphi(u)| < |u|].$$

2.2. $\varphi(x, i) = \varphi(\varphi(x), i)$; $\varphi(xyz, i) = \varphi(x\varphi(y)z, i)$.

2.3. $f(a_1^{\delta(f)}) = (b_1^{\rho(f)}) \Rightarrow \varphi(x f a_1 \dots a_{\delta(f)} y, i) = \varphi(x b_1 \dots b_{\rho(f)} y, i)$.

2.4. $\varphi(x (y, 1) (y, 2) \dots (y, m) z, i) = \varphi(xyz, i)$.

Define now an $(m+1, m)$ -operation $[]$ on B^\wedge by

$$[u_1^{m+1}] = (v_1^m) \Leftrightarrow (\forall i \in N_m) v_i = \varphi(u_1^{m+1}, i), \text{ where } u_r, v_s \in B^\wedge.$$

2.5. $(B^\wedge; [])$ is an $(m+1, m)$ -semigroup.

Proof: Let $u_r, v_s \in B^\wedge$. Then we have:

$$\begin{aligned} [[u_1^{m+1}] u_{m+2}] &= (v_1^m) \Leftrightarrow v_i = \varphi(\varphi(u_1^{m+1}, 1) \dots \varphi(u_1^{m+1}, m) u_{m+2}, i) \\ &= \varphi((u_1^{m+1}, 1) \dots (u_1^{m+1}, m) u_{m+2}, i) \\ &= \varphi(u_1^{m+2}, i) = \varphi(u_1 (u_2^{m+2}, 1) \dots (u_2^{m+2}, m), i) \\ &= \varphi(u_1 \varphi(u_2^{m+2}, 1) \dots \varphi(u_2^{m+2}, m), i) \text{ for all } i \in N_m \\ &\Leftrightarrow [u_1 [u_2^{m+2}]] = (v_1^m), \end{aligned}$$

which implies that $(B^\wedge; [])$ is an $(m+1, m)$ -semigroup.

We have to show that (0.4) is satisfied.

Let f, a_r, b_s be such that $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$ and $\rho(f) > m$.

Then

$$[f a_1^{\delta(f)}] = (u_1^m) \Leftrightarrow (\forall i \in N_m) u_i = \varphi(f a_1^{\delta(f)}, i) = (b_1^{\rho(f)}, i) \Leftrightarrow [b_1^{\rho(f)}] = (u_1^m).$$

Conversely, $[f a_1^{\delta(f)}] = [b_1^{\rho(f)}] = (u_1^m)$ implies

$$u_i = \varphi(f a_1^{\delta(f)}, i) = (f(a_1^{\delta(f)}), i), \quad u_i = \varphi(b_1^{\rho(f)}, i) = (b_1^{\rho(f)}, i)$$

for all $i \in N_m$, i.e. $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$.

Suppose now that $\rho(f) = m$, $f(a_1^{\delta(f)}) = (b_1^m)$. Then

$$[f a_1^{\delta(f)}] = (u_1^m) \Leftrightarrow u_i = \varphi(f a_1^{\delta(f)}, i) = b_i, \text{ for all } i \in N_m \Leftrightarrow [f a_1^{\delta(f)}] = (b_1^m).$$

This completes the proof of the Theorem, since $(B^\wedge; [\])$ is the desired $(m+1, m)$ -semigroup.

3. Here we make a few remarks.

3.1. The fact that an $(m+1, m)$ -semigroup $(Q; [\])$ induces an $(m+k, m)$ -semigroup $(Q; [\]^k)$ implies that the following generalization is a corollary of the Theorem.

Theorem 1. Let $(A; F)$ be a v.v.a. and let m, k be positive integers. Suppose that for every $f \in F$ there exist integers s_f, r_f such that

$$1 + \delta(f) = m + ks_f, \rho(f) = m + kr_f, r_f \geq 0, s_f \geq 1.$$

Then, there is an $(m+k, m)$ -semigroup $(Q; [\])$ and a mapping $\alpha: f \mapsto \mathbf{f}$ from F into Q such that $A \subseteq Q$ and (0.4) is satisfied for any $a_i, b_j \in A, f \in F$.

3.2. If $(A; F)$ is a usual universal algebra, i.e. if $\rho(f) = 1$ for any $f \in F$, then $m = 1$ is the unique positive integer such that $\rho(f) \geq m$, and in this case the result of the Theorem is the well known Cohn-Rebane's theorem ([1], [4]). That is why we call our Theorem "Cohn-Rebane theorem for v.v.a.". The original Cohn-Rebane's Theorem can be "translated" for v.v.a. in the following way.

Theorem 2. If $(A; F)$ is a v.v.a. then there is a semigroup $(Q; \cdot)$ and a mapping $\alpha: f \mapsto f_1 f_2 \cdot \dots \cdot f_{\rho(f)}$ from F into Q^+ such that $A \subseteq Q, f_i \in Q$ and

$$f(a_1^{\delta(f)}) = (b_1^{\rho(f)}) \Leftrightarrow b_i = f_i a_1 \dots a_{\delta(f)}$$

for any $f \in F, a_i, b_j \in A$.

We note that if $m \geq 2$ or $(\exists f \in F) \rho(f) \geq 2$, then the assertion of our Theorem is not a consequence of Theorem 2.

3.3. The $(m+1, m)$ -semigroup $(B^\wedge; [\])$ obtained in the proof of the Theorem has the following universal property. If $(P; [\]')$ is an $(m+1, m)$ -semigroup and $\alpha': f \mapsto \mathbf{f}'$ a mapping from F into P such that $A \subseteq P$ and

$$f(a_1^{\delta(f)}) = (b_1^{\rho(f)}) \Leftrightarrow [f' a_1^{\delta(f)}]' = [b_1^{\rho(f)}]', \quad (0.4')$$

then there exists a unique homomorphism $\xi: (B^\wedge; [\]) \rightarrow (P; [\]')$ such that $\xi(\mathbf{f}) = (\mathbf{f}'), \xi(a) = a$, for any $f \in F, a \in A$.

3.4. Throughout the paper, it was implicitly assumed that $F \neq \emptyset$, and if we allow the case $F = \emptyset$, then the $(m+1, m)$ -semigroup $(B^\wedge; [\])$ obtained in the proof of the Theorem would be the free $(m+1, m)$ -semigroup with a basis A . A convenient description of free vector valued semigroups is given in paper [2], and we would like to note that in the above proof of the Theorem some ideas from that paper are used.

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ТЕОРЕМАТА НА КОН-РЕБАНЕ ЗА ВЕКТОРСКО ВРЕДНОСНИ АЛГЕБРИ

(Резиме)

Во работава се докажува следнава

Теорема: Нека (A, F) е векторско вредносна алгебра и m позитивен цел број. Постои $(m+1, m)$ -полугрупа $(Q, [\])$ и пресликување $\alpha: f \mapsto \mathbf{f}$, така што $A \subseteq Q$ и

$$f(a_1^{\delta(f)}) = (b_1^{\rho(f)}) \Leftrightarrow [f a_1^{\delta(f)}] = [b_1^{\rho(f)}]$$

за секои $a_i, b_j \in A, f \in F$.