ON GROUPOIDS WITH THE IDENTITY $x^2y^2 = xy$

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A b s t r a c t. The variety $\mathcal{U}$ of groupoids which satisfy the identity $x^2y^2 = xy$ is considered.

0. *Introduction*

The main object of the paper, which consists of four sections, are free groupoids in the variety $\mathcal{U}$.

A description of free groupoids in $\mathcal{U}$ is given in the first section. In fact, first we show that the reduction $x^2y^2 \to xy$ is not convenient for obtaining corresponding canonical free groupoids in $\mathcal{U}$, and afterwards we show that the pair of reductions: $xy^2 \to xy$, $x^2y \to xy$ has the desired properties.[1)]

The class of $\mathcal{U}$ - injective groupoids (which is a proper subclass of $\mathcal{U}$, and properly contains the class of $\mathcal{U}$ - free [2)] groupoids) is considered in Section 2. Some characterizations of $\mathcal{U}$ - free groupoids within the class of $\mathcal{U}$ - injective groupoids are also given. It is shown that neither of the classes of $\mathcal{U}$ - injective groupoids, $\mathcal{U}$ - free groupoids is hereditary. Namely, a subgroupoid $\boldsymbol{Q}$ of a $\mathcal{U}$ - injective ($\mathcal{U}$ - free) groupoid $\boldsymbol{H}$ is $\mathcal{U}$-injective ($\mathcal{U}$ - free) if each idempotent in $\boldsymbol{Q}$ is a square of a nonidempotent in $\boldsymbol{Q}$.

Some characterizations of $\mathcal{U}$ - free subgroupoids of a $\mathcal{U}$ - free groupoid are given in Section 3. It is also shown that a $\mathcal{U}$ - free groupoid with rank 2 admits subgroupoids with infinite rank.

In Section 4, as a corollary of Evans' results ([3] and [4]) we obtain that the word problem is solvable in $\mathcal{U}$.

We note that the results of this paper have corresponding analogies in the variety of commutative $\mathcal{U}$ - groupoids, but we do not translate them explicitely.

1. *Canonical $\mathcal{U}$ - groupoids*

Throughout the paper we denote by $\boldsymbol{F} = (F, \cdot)$ a free groupoid (in the variety of all groupoids) with a given basis B. It is well-known (for example [1; I.1]) that the following two properties characterise $\boldsymbol{F}$.

a) $ab = cd \;\Rightarrow\; a = c,\; b = d$.

(Any groupoid with this property is said to be *injective*.)

b) *B is the set of primes in $\boldsymbol{F}$ and it generates $\boldsymbol{F}$.*

(An element $c \in G$ is *prime* in a groupoid $\boldsymbol{G} = (G, \cdot)$ iff $c \neq xy$, for all $x, y \in G$.)

The *norm* in $\boldsymbol{F}$ is the homomorphism $x \mapsto |x|$ from $\boldsymbol{F}$ into the additive groupoid $(\mathbb{N}, +)$ of positive integers which extends the mapping $B \to \{1\}$. Thus:

$$|b| = 1, \quad |uv| = |u| + |v| ,$$

for $b \in B$, $u, v \in F$.

Let $\mathcal{V}$ be a variety of groupoids such that there exists a groupoid $\boldsymbol{R} = (R, *)$ with the following properties:

c) $B \subseteq R \subseteq F \;\;\&\;\; (\forall\, t, u \in F)\ \{tu \in R \;\Rightarrow\; t, u \in R \;\;\&\;\; t * u = tu\}$.

d) *$\boldsymbol{R}$ is free in $\mathcal{V}$ with a basis B.*

Then we say that $\boldsymbol{R}$ is a *$\mathcal{V}$-canonical groupoid.*

Below we are looking for a $\mathcal{U}$-canonical groupoid with a given basis B.

The identity

$$x^2y^2 = xy \tag{1.1}$$

(which is the axiom of $\mathcal{U}$) suggests to consider the following groupoid $\boldsymbol{S} = (S, \bullet)$ as a candidate for a $\mathcal{U}$- canonical groupoid with the basis B. As the carrier of $\boldsymbol{S}$, choose the subset S of all the elements of F without subwords of the form $\alpha^2\beta^2$, and define the operation $\bullet$ on S in the following way:

1) Here we use the following usual abbreviations: $x^2y^2 = (xx)(yy)$, $xy^2 = x\,(yy)$, $x^2y = (xx)y$

2) "$\mathcal{U}$- free groupoid" means "groupoid free in $\mathcal{U}$".

$$(\forall\, t, u \in S) \qquad t \bullet u = \begin{cases} tu & \text{if } tu \in S \\ \alpha\beta & \text{if } t = \alpha^2,\ u = \beta^2 \end{cases}$$

However, although $\boldsymbol{S}$ is a well-defined groupoid (with the least generating subset B), which satisfies the condition c), $\boldsymbol{S}$ does not belong to I. (Namely, if $b \in B$, then b, $b^2 \in S$, but $(b^2 \bullet b^2) \bullet (b \bullet b) = (b \bullet b) \bullet (b \bullet b) = b^2 \neq b^2 b = b^2 \bullet b$.)

The following property of $\mathcal{U}$ will provide a system of axioms of $\mathcal{U}$ which will enable us to obtain a $\mathcal{U}$-canonical groupoid.

1.1. *A groupoid* $\boldsymbol{G} = (G, \cdot)$ *belongs to* $\mathcal{U}$ *iff it satisfies the following identities:*

$$x^2 y = xy, \qquad xy^2 = xy. \tag{1.2}$$

Proof. If $\boldsymbol{G}$ satisfies (1.2), then for any $a, b \in G : ab = ab^2 = a^2 b^2$, i.e. $\boldsymbol{G} \in \mathcal{U}$. Conversely, if $\boldsymbol{G} \in \mathcal{U}$, then:

$$a^2 b^2 = a^2 (bb) = a^2 (b^2 b^2) = a^2 (b^2)^2 = ab^2.$$

Similarly: $a^2 b^2 = a^2 b$. □

1.2. *If* $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}$ *and* $a \in G$, *then:*

$$(a^2)^2 = a^2 = aa^2 = a^2 a.$$

(*Therefore, for each* $a \in G$, a^2 *is an idempotent.*) □

Remark. Two varieties of groupoids, $\mathcal{U}_r$ and $\mathcal{U}_l$, defined by the axioms $x^2 y = xy$ and $xy^2 = xy$ respectively, are investigated in [2]. By **1.1**, it follows that $\mathcal{U} = \mathcal{U}_r \cap \mathcal{U}_l$. As a corollary, one obtains that, for any $k, m, s \in \mathbb{N}$, where $k \geq 2$, the following identities are true in $\mathcal{U}$:

$$x^k = x^2, \quad x^m y = xy^s = xy \tag{1.3}$$

Here, for a given $n \in \mathbb{N}$, x^n denotes any of $\dfrac{(2n-2)!}{n!(n-1)!}$ possible n-th powers.

Now denote by R the set of all elements $u \in F$ such that u has not a subword of the form $\alpha\beta^2$ neither of the form $\alpha^2\beta$.

Using the transformation $t \mapsto \underline{t}$ in F, defined as follows:

$$\underline{t} = \begin{cases} t & \text{if } (\forall \alpha \in F)\ t \neq \alpha^2 \\ \alpha & \text{if } \ t = \alpha^2 \end{cases}$$

we obtain the following characterization of R:

1.3. R *is the least subset of* F *such that* $B \subseteq R$ *and satisfies the following statement:*

$$(\forall\, t, u \in F) \{ tu \in R \Leftrightarrow [\, t, u \in R \ \ \& \ \ (\underline{t} = t,\ \underline{u} = u)] \}. \ \square$$

The following implications are clear:

$$t \in R \ \Rightarrow \ u = \underline{t} \in R \ \ \& \ \ \underline{u} = u \tag{1.5}$$

$$t, u \in R \ \Rightarrow \ \underline{t}\,\underline{u} \in R \tag{1.6}$$

Now we will show the following

Theorem 1.4. *If the operation* $*$ *on* R *is defined by*

$$(\forall\, t, u \in R) \quad t * u = \underline{t}\,\underline{u}\ , \tag{1.7}$$

then $\boldsymbol{R} = (R, *)$ *is a* $\mathcal{U}$-*canonical groupoid with the basis* B.

Proof. 1) From (1.5) and (1.6) it follows that $*$ is a well-defined operation on R, i.e. that $\boldsymbol{R}$ is a groupoid, and it is clear that B is the least generating set of $\boldsymbol{R}$.

2) By (1.4), **1.3** and (1.7), the condition c) is satisfied.

3) Let $t, u \in R$ and $v = \underline{t}$, $w = \underline{u}$. Then:

$$(t*t)*(u*u) = (\underline{t})^2 * (\underline{u})^2 = v^2 * w^2 = v * w = t * u, \text{ i.e. } \boldsymbol{R} \in \mathcal{U}.$$

4) Let $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}$, $\lambda : B \to G$ be any mapping and $\varphi : F \to G$ be the homomorphism from $\boldsymbol{F}$ into $\boldsymbol{G}$ which is an extension of λ. If $t, u \in R$, then:

$$\varphi(t * u) = \varphi(\underline{t}\,\underline{u}) = \varphi(\underline{t})\, \varphi(\underline{u}) = \varphi(\underline{t})^2\, \varphi(\underline{u})^2 = \varphi(tu).$$

Therefore, the restriction ψ of φ on R is a homomorphism from $\boldsymbol{R}$ into $\boldsymbol{G}$, and ψ extends λ. □

Bellow we assume that $\boldsymbol{H} = (H, \cdot)$ is a $\mathcal{U}$- free groupoid with a given basis B, and therefore it is isomorphic to the above obtained $\mathcal{U}$ - canonical groupoid, which implies that the following statements hold.

1.5. *The basis of* $\boldsymbol{H}$ *coincides with the set of primes in* $\boldsymbol{H}$. □

1.6. *If* $a \in HH$, *then there exists a unique pair* $(b, c) \in H^2$, *such that* $a = bc$, *and neither of* b, c *is idempotent. Then,* a *is idempotent iff* $b = c$. □

1.7. *There is a mapping* $a \mapsto |a|$ *from* H *into* $\mathbb{N}$ *such that:* $|bc| \leq |b| + |c|$, *where the equality holds iff neither of* b, c *is idempotent.* □

1.8. *If* $B = \{b\}$ *is a singleton, then* $H = \{e, b\}$, *where* $e \neq b$, *and* $xy = b$ *in* $\boldsymbol{H}$, *for any* $x, y \in H$. □

1.9. *If* B *contains at least two distinct elements, then* H *is infinite.*

Proof. Assume that $B = \{a, b\}$, where $a \neq b$. Consider the $\mathcal{U}$- canonical groupoid $\boldsymbol{R}$ with basis B, and define the subset $C = \{c_k \mid k \in \mathbb{N}\}$:

$$c_1 = ab, \quad c_{k+1} = c_k b. \tag{1.8}$$

Then $B \subset R$ and $c_m = c_n \Leftrightarrow \quad m = n$. □

2. *Canonical* $\mathcal{U}$*–injective groupoids*

A $\mathcal{U}$- groupoid $\boldsymbol{H} = (H, \cdot)$ is said to be $\mathcal{U}$- *injective* iff it satisfies the following condition:

(inj) For every element $a \in HH$, there is a unique pair of *nonidempotents* $b, c \in H$ (i.e. $b^2 \neq b$, $c^2 \neq c$), such that $a = bc$. In that case, $b = c$ iff a is an idempotent. (Then we say that (b, c) is *the pair of divisors* of a in $\boldsymbol{H}$, and write $b \mid a$, $c \mid a$.)

From **1.3** and **1.6** we obtain:

2.1. *If* $\boldsymbol{H} = (H, \cdot)$ *is a* $\mathcal{U}$*- free groupoid, then* $\boldsymbol{H}$ *is* $\mathcal{U}$*- injective and there is a maping* $a \mapsto |a|$ *from* H *into* $\mathbb{N}$ *such that:*

(b, c) *is the pair of divisors of* a *implies that* $|a| = |b| + |c|$. □

Below we assume that $\boldsymbol{H} = (H, \cdot)$ is a given $\mathcal{U}$- injective groupoid, and N, I are the sets of nonidempotents, idempotents in $\boldsymbol{H}$, respectively. Also the fact that $(a^2)^2 = a^2$, for all $a \in H$ (see **1.2**), implies the following property which gives a convenient description of $\mathcal{U}$- injective groupoids.

2.2. *If* $\boldsymbol{H} = (H, \cdot)$ *is a groupoid, then the following two conditions are equivalent.*

(i) $\boldsymbol{H}$ *is* $\mathcal{U}$*- injective.*

(ii) There exist two nonempty disjoint subsets N, I *of* $\boldsymbol{H}$, *a bijection* $\varphi : N \to I$, *and an injection* $\psi : N^2 \setminus \{(a, a) \mid a \in N\} \to N$, *such that* $H = N \cup I$ *and:*

$$\begin{aligned} \varphi(a)\, \varphi(a) = a\varphi(a) = \varphi(a)\, a = a\, a = \varphi(a), \\ \varphi(a)\, \varphi(b) = a\varphi(b) = \varphi(a)\, b = a\, b = \varphi(a, b), \end{aligned} \tag{2.1}$$

for any $a, b \in N$, *and* $a \neq b$.

Proof. If $\boldsymbol{H}$ is $\mathcal{U}$-injective, and I, N are the sets of idempotents, nonidempotents in $\boldsymbol{H}$, respectively, then the mapping $\varphi : a \mapsto a^2$ is a bijection from N onto I, and ψ: $(a, b) \mapsto ab$ is an injection from $N^2 \setminus \{(a, a) \mid a \in N\}$ into N such that the equations (2.1) hold.

Conversely, if $\boldsymbol{H} = (H, \cdot)$ satisfies the conditions stated in *(ii)*, then I, N are the sets of idempotents, nonidempotents in $\boldsymbol{H}$, respectively. From (2.1), it follows that $\boldsymbol{H} \in \mathcal{U}$. Let $a \in HH$; if $a \in I$ and $b = \varphi^{-1}(a)$, then (b, b) is the pair of divisors of a in $\boldsymbol{H}$; and if $a \in N$, there exists a unique pair $(b, c) \in N^2$ and $a = bc = \psi(b, c)$, and thus (b, c) is the pair of divisors of a in $\boldsymbol{H}$. □

(Below we denote by $(N, I; \varphi, \psi)$ the $\mathcal{U}$- injective groupoid defined in **2.2** *(ii)*. Namely, from **2.2** it follows that N, I, φ and ψ are uniquely determined by the given $\mathcal{U}$- injective groupoid $\boldsymbol{H}$.)

As a corollary from **2.2** it follows that, if $\boldsymbol{H}$ is a finite $\mathcal{U}$- groupoid, and $|N| = n$, then $n^2 - n \leq n$, i.e. $1 \leq n \leq 2$. Therefore, we obtain the following *description of finite* $\mathcal{U}$*- injective groupoids*.

2.3. *A* I *- injective groupoid* ***H*** *is finite iff it is isomorphic to one of the groupoids defined by the following multiplication tables:*

$\cdot$	i	a
i	i	i
a	i	i

$\cdot$	i	j	a	b
i	i	b	i	b
j	a	j	a	j
a	i	b	i	b
b	a	j	a	j

As a corollary of **1.6** and **2.3** we obtain:

2.4. *The class of* $\mathcal{U}$ *- free groupoids is a proper subclass of the class of* $\mathcal{U}$*- injective groupoids.* □

The following statement is a description of $\mathcal{U}$- free groupoids within the class of $\mathcal{U}$- injective groupoids.

Theorem 2.5. *If* $\boldsymbol{H}=(H,\cdot)$ *is a* $\mathcal{U}$*- injective groupoid, then the following conditions are equivalent.*

(i) ***H*** *is* $\mathcal{U}$*-free.*

(ii) There is a mapping $a\mapsto|a|$ *from* H *into* $\mathbb{N}$*, such that:*

if (b,c) *is the pair of divisors of* a*, then* $|a|=|b|+|c|$*.*

(iii) The set of primes in ***H*** *generates* ***H****.*

Proof. By **1.6** and **1.7**, *(i)* $\Rightarrow$ *(ii)*.

Assuming that *(ii)* holds, by induction, it can be easily shown that B generates ***H***, i.e. we obtain: *(ii)* $\Rightarrow$ *(iii)*.

Finally, assume that the set B of primes in ***H*** generates ***H***. B is nonempty, for if $a\in H$, then $\{a^2\}$ is a proper subgroupoid of ***H***. Let $\boldsymbol{R}=(R,*)$ be the $\mathcal{U}$- canonical groupoid with the basis B, and let $\varphi:R\to H$ be the homomorphism from ***R*** into ***H*** which extends the embedding $B\to H$. Then φ is surjective, for B generates the both groupoids ***R*** and ***H***. Thus we have to show that φ is injective as well, and this can be easily shown by induction using the mapping $u\mapsto|u|$. □

The fact that, for any $a\in H$ (where $\boldsymbol{H}=(H,\cdot)$ is a $\mathcal{U}$- injective groupoid), the set $\{a^2\}$ is a subgroupoid (of ***H***) which is not $\mathcal{U}$- injective, by **2.4**, implies the following proposition:

2.6. *Neither of the classes of* $\mathcal{U}$*- injective,* $\mathcal{U}$*-free groupoids is hereditary.* □

In the next section we will use the following description of the $\mathcal{U}$- injective subgroupoids, which is a corollary from the proof of **2.2.**

2.7. *Let* $\boldsymbol{H}=(N,I;\varphi,\psi)$ *be a* $\mathcal{U}$*- injective groupoid and* $\boldsymbol{Q}=(Q,\cdot)$ *a subgroupoid of* ***H****.* ***Q*** *is* $\mathcal{U}$*- injective iff* $a\in Q\cap I\ \Rightarrow\ \varphi^{-1}(a)\in Q$. □

3. *Subgroupoids of* $\mathcal{U}$*-free groupoids*

Below we assume that ***H*** is a $\mathcal{U}$- free groupoid with the basis B, and ***Q*** a subgroupoid of ***H***. Therefore (by **2.5** and **2.1**), ***H*** is $\mathcal{U}$- injective, and there exists a mapping $a\mapsto|a|$ from H into $\mathbb{N}$, such that: if (b,c) is the pair of divisors of $a\in HH$, then $|a|=|b|+|c|$. Thus, for any $a\in Q$, $|a|\in\mathbb{N}$, but ***Q*** is not necessarily $\mathcal{U}$- injective, as we saw in **2.6**. We will divide the set of idempotents in ***Q*** into two parts: regular and singular. Namely, an idempotent $a\in Q$ is said to be *regular* in ***Q*** iff the divisor b belongs to Q; in the contrary, a is said to be *singular* in ***Q***. The set of singular idempotents in ***Q*** will be denoted by S, and the set of primes in ***Q*** by P.

We will prove the following lemma:

3.1. $P\cup S$ *is the least generating set for* ***Q****.*

Proof. If $T\subseteq Q$, then by $\langle T\rangle$ will be denoted the subgroupoid of ***Q*** generated by T. The following relations are clear:

$$p\in P\ \Rightarrow\ p\notin\langle Q\setminus\{p\}\rangle \tag{3.1}$$

$$a\in S\ \Rightarrow\ a\notin\langle Q\setminus\{a\}\rangle \tag{3.2}$$

From (3.1) and (3.2) it follows that $P\cup S$ is a subset of every generating set of ***Q***. So, it remains to show that $P\cup S$ generates ***Q***.

Let $\alpha \in Q$. We will show that $\alpha \in \langle P \cup S \rangle$ by induction on $|\alpha|$. First, if $|\alpha| = \min\{|a| \mid a \in Q\}$, then $\alpha \in P \cup S$. Assume that

$$\alpha \in Q \;\;\&\;\; |\alpha| \leq k \;\Rightarrow\; \alpha \in \langle P \cup S \rangle .$$

Let $\beta \in Q$ be such that $|\beta| = k+1$, $\beta \notin P \cup S$. Then there exist $\gamma, \delta \in Q$ such that $\beta = \gamma\delta$. If γ and δ are nonidempotents, then $|\gamma| + |\delta| = |\beta|$, and so $\gamma, \delta \in \langle P \cup S \rangle$, which implies that $\beta \in \langle P \cup S \rangle$. The same conclusion arises also in the case $\gamma, \delta \notin S$. It is clear that: $\gamma, \delta \in S \Rightarrow \beta = \gamma\delta \in \langle P \cup S \rangle$. Therefore we can assume that, for example, $\gamma \in Q \setminus S$, $\delta \in S$. Then we may consider that γ is a nonidempotent, and this implies $|\gamma| < k$, i.e. $\gamma \in Q$. □

As a corollary of the above lemma (and also **2.7**), we obtain the following *characterization of* $\mathcal{U}$*-free subgroupoids of a* $\mathcal{U}$*-free groupoid.*

Theorem 3.2. *Let* $\boldsymbol{H}$ *be a* $\mathcal{U}$*-free groupoid. If* $\boldsymbol{Q}$ *is a subgroupoid of* $\boldsymbol{H}$*, then the following conditions are equivalent:*

a) $\boldsymbol{Q}$ *is* $\mathcal{U}$*-injective.*

a$'$) *Every idempotent in* $\boldsymbol{Q}$ *is regular.*

b) $\boldsymbol{Q}$ *is* $\mathcal{U}$*-free.*

c) *The set* P *of primes in* $\boldsymbol{Q}$ *generates* $\boldsymbol{Q}$*.*

Proof. *a)* $\Leftrightarrow$ *a')*, by **2.7** and the definition of regular idempotents.

Let $\boldsymbol{Q}$ be $\mathcal{U}$-injective. For any $a \in QQ$, there exists a uniquely determined pair (b, c) of divisors of a in $\boldsymbol{Q}$. Then, (b, c) is the pair of divisors of a in $\boldsymbol{H}$, and : $|b| + |c| = |a|$. Therefore, by Th.2.5, $\boldsymbol{Q}$ is $\mathcal{U}$-free. Thus: *a)* $\Rightarrow$ *b)*. Also *b)* $\Rightarrow$ *c)* by Th.2.5. Finally, from *c)* and **3.1**, we obtain that $S = \varnothing$, i.e. a$'$) holds.□

The situation with the ranks of $\mathcal{U}$-free subgroupoids of $\mathcal{U}$-free groupoids is similar as for the varieties $\mathcal{U}_r$ and $\mathcal{U}_l$. Namely, if $\boldsymbol{H}$ is a $\mathcal{U}$-free groupoid with rank 1, then H has only two elements, and $\boldsymbol{H}$ is the unique $\mathcal{U}$-free subgroupoid of $\boldsymbol{H}$. $\mathcal{U}$-free groupoids with rank greater then 1, acording to **1.9**, have the following property:

3.3. *If* $\boldsymbol{H}$ *is a* $\mathcal{U}$*-free groupoid with rank 2, then there is a* $\mathcal{U}$*-free subgroupoid* $\boldsymbol{Q}$ *of* $\boldsymbol{H}$ *with infinite rank.* □

4. *Word problem for* $\mathcal{U}$

As a corollary of an Evans' result in [3], below we will obtain the following

Theorem 4.1. *The word problem for the variety* $\mathcal{U}$ *is solvable.*

First recall ([1; I.1]) that $\boldsymbol{G} = (G, \cdot)$ is a *multigroupoid* if $(a, b) \mapsto a \cdot b$ is a mapping from G^2 into the collection of the subsets (including the empty one) of G. If, for any $a, b \in G$, $a \cdot b$ contains at most one element, then $\boldsymbol{G}$ is caled a *halfgroupoid*. Thus, a *groupoid* may be considered as a special kind of a halfgroupoid which satisfies the condition:

$$(\forall\, a, b \in G) \;\; a \cdot b \neq \varnothing \;,$$

where the equation $c = \{c\}$ is assumed. A halfgroupoid $\boldsymbol{G} = (G, \circ)$ is called a *subhalfgroupoid* of a halfgroupoid $\boldsymbol{H} = (H, \cdot)$ iff $G \subseteq H$ and:

$$(\forall\, a, b \in G) \;\; [\, a \cdot b \neq \varnothing \;\Rightarrow\; a \circ b = a \cdot b \,].$$

Finally, we say that a halfgroupoid $\boldsymbol{G}$ is a $\mathcal{U}$*-halfgroupoid* iff the following condition holds:

$$(\forall\, a, b, c \in G) \;\; [\, a^2 \neq \varnothing \;\Rightarrow\; ab = a^2 b \;\;\&\;\; ca = ca^2 \,]. \tag{4.1}$$

A special case of the Evans' Theorem in [3; p.68] is the following proposition.

4.2. *If each* $\mathcal{U}$*-halfgroupoid is a subhalfgroupoid of a* $\mathcal{U}$*-groupoid, then the word problem is solvable for the variety* $\mathcal{U}$*.* □

Thus, Th.4.1 is a corollary of **4.2** and the following:

Lemma 4.3. *Let* $(G, \cdot)$ *be a* $\mathcal{U}$*-halfgroupoid and* e *a fixed element in* G*. Define a groupoid* $(G, \circ)$ *as follows:*

$$x \cdot y \neq \varnothing \;\Rightarrow\; x \circ y = x \cdot y, \tag{4.2}$$

$$x^2 = \varnothing \;\Rightarrow\; x \circ x = x, \tag{4.3}$$

$$x \neq y \;\;\&\;\; x \cdot y = \varnothing \;\Rightarrow\; x \circ y = e. \tag{4.4}$$

Then, $(G, \circ)$ *is a* $\mathcal{U}$*- groupoid and* $(G, \cdot)$ *is its* $\mathcal{U}$*- subhalfgroupoid.*

Proof. Let $a, b \in G$. If $a^2 = \varnothing$, then by (4.3):

$$(a \circ a) \circ b = a \circ b.$$

If $a^2 \neq \varnothing$, then by (4.1) and (4.2):

$$(a \circ a) \circ b = a^2 \circ b = a^2 b = ab = a \circ b, \text{ in the case } a \cdot b \neq \varnothing, \text{ and}$$

$$(a \circ a) \circ b = e = a \circ b, \text{ if } ab = \varnothing.$$

Similarly, one shows that $b \circ (a \circ a) = b \circ a$.

Thus, $(G, \circ)$ is a $\mathcal{U}$- groupoid. □

The notion of a $\mathcal{U}$- halfgroupoid depends essentially on the set of axioms of $\mathcal{U}$ (which follows also by the Evans' general note in [3; p.66]). Namely, the definition of a $\mathcal{U}$ - halfgroupoid given by (4.1) assumes the axioms (1.2). If one chooses (1.1) as the axiom of $\mathcal{U}$, then a halfgroupoid $\boldsymbol{G} = (G, \cdot)$ would be called a $(\mathcal{U}, 1)$ - *halfgroupoid* if:

$$(\forall\, a, b \in G) \quad [\, a^2 \neq \varnothing,\ b^2 \neq \varnothing \ \Rightarrow\ a^2 b^2 = ab \,]. \qquad (4.1')$$

There are $(\mathcal{U}, 1)$ - halfgroupoids which are not subhalfgroupoids of $\mathcal{U}$- groupoids. Such a subhalfgroupoid is defined by the following multiplication table:

	a	b	c
a	b	b	a
b	b	b	b
c	$\varnothing$	$\varnothing$	$\varnothing$

Therefore, L. 4.3 would not be true, and Th.4.1 would not be a corollary of [3].

However, it is easy to show that the problem: whether a given finite $(\mathcal{U}, 1)$ - halfgroupoid $\boldsymbol{G}$ is a subhalfgroupoid of a $\mathcal{U}$- groupoid is solvable. Therefore, if one chooses (1.1) as the axiom of $\mathcal{U}$, Th.4.1 is a corollary of the main Evans' result in [4; p.76] .

Finally, we note that an algorithm for solving the word problem for the varieties of more general kind is formulated explicitely in [3]. This algorithm is simpler for the variety $\mathcal{U}$, because here it is possible to use the canonical $\mathcal{U}$- groupoid $\boldsymbol{R}$ instead of $\boldsymbol{F}$.

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ЗА ГРУПОИДИТЕ СО ИДЕНТИТЕТОТ $x^2 y^2 = xy$

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Предмет на статијава е многубразието $\mathcal{U}$ од групоиди што го задоволуваат идентитетот $x^2 y^2 = xy$. Прво се дава опис на $\mathcal{U}$ - слободни (т.е. слободни во $\mathcal{U}$) групоиди (Th.1.4), а потоа се наведуваат некои нивни карактеризации во рамките на класата $\mathcal{U}$ - инјективни групоиди (Th.2.5). Притоа се покажува дека ниедна од спомeнатите класи ($\mathcal{U}$- слободни, $\mathcal{U}$- инјективни групоиди) не е наследна (Prop. 2.6). Натаму се дава карактеризација на $\mathcal{U}$- слободни подгрупоиди од $\mathcal{U}$- слободни групоиди (Th.3.2), а на крајот, како последица од главниот резултат на Еванс од [3], се добива дека проблемот на зборови е решлив во $\mathcal{U}$.