FREE GROUPOIDS WITH $x^n = x$

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Abstract: We give a description of the class of free objects in the variety $\mathcal{U}_n$ of groupoids which satisfy the law $x^n = x$. In Section **0** we state necessary preliminary definitions and the main results of the paper, which are shown in Sections **1-3**. Corresponding results concerning free objects in commutative $\mathcal{U}_n$-groupoids are stated in Section **4**.

0. INTRODUCTION

A *groupoid* is an algebra $\boldsymbol{G} = (G, \cdot)$, where $(x, y) \mapsto xy$ is a binary operation on G. Notions as subgroupoids, homomorphisms and variety of groupoids have usual meanings. If $\mathcal{U}$ is a variety of groupoids, then "$\boldsymbol{G}$ is a $\mathcal{U}$-groupoid", "$\boldsymbol{G}$ is a $\mathcal{U}$-free groupoid", "$\boldsymbol{G}$ is a free groupoid", means "$\boldsymbol{G} \in \mathcal{U}$", "$\boldsymbol{G} \in \mathcal{U}$ and $\boldsymbol{G}$ is free in $\mathcal{U}$", "$\boldsymbol{G}$ is free in the variety of all groupoids", respectively.

In this paper we assume that n is a given integer such that $n \geq 2$, and $\mathcal{U}_n$ is the variety of groupoids which satisfies the law $x^n = x$, where:

$$x^1 = x, \qquad x^k x = x^{k+1}.$$

Let $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}_n$. An element $a \in \boldsymbol{G}$ is called *$\mathcal{U}_n$-prime* in $\boldsymbol{G}$ iff the following implication holds in $\boldsymbol{G}$:

$$a = bc \;\Rightarrow\; b = c^{n-1}. \tag{0.1}$$

Thus, $a \in \boldsymbol{G}$ is *not* *$\mathcal{U}_n$-prime* in $\boldsymbol{G}$ iff there exist $b, c \in \boldsymbol{G}$ which satisfy the following statement:

$$a = bc \;\;\&\;\; b \neq c^{n-1}. \tag{0.1$'$}$$

Then we say that b and c are *$\mathcal{U}_n$-divisors of a in $\boldsymbol{G}$*. A (finite or infinite) sequence a_1, a_2, … of elements in $\boldsymbol{G}$ is called a *$\mathcal{U}_n$-divisor chain in $\boldsymbol{G}$* iff $a_{k+1}|a_k$ [1], for all $k = 1,2,\ldots$; we assume that, for any $a \in \boldsymbol{G}$, the (one-element) sequence a is a $\mathcal{U}_n$-divisor chain in $\boldsymbol{G}$. And, $\boldsymbol{G}$ is *$\mathcal{U}_n$-injective* iff, for any $a, b, c, d \in \boldsymbol{G}$, the following implication holds:

$$ab = cd \;\;\&\;\; a \neq b^{n-1} \;\;\&\;\; c \neq d^{n-1} \;\Rightarrow\; a = c \;\;\&\;\; b = d. \tag{0.2}$$

Now we state the main results.

Theorem 1. *Let B be a non-empty set and $\boldsymbol{F} = (F, \cdot)$ a free groupoid with the basis B. Let R, a subset of F, and $\boldsymbol{R} = (R, *)$ be defined as follows:*

$B \subseteq R$ *and*

$$u, v \in F \;\Rightarrow\; \{uv \in R \Leftrightarrow u, v \in R \;\;\&\;\; u \neq v^{n-1}\}, \tag{0.3}$$

$$u, v \in R \;\Rightarrow\; u * v = \begin{cases} uv, & \text{if } uv \in R \\ v, & \text{if } u = v^{n-1} \end{cases} \tag{0.4}$$

Then $\boldsymbol{R}$ is a $\mathcal{U}_n$-free groupoid and B is the unique $\mathcal{U}_n$-basis of $\boldsymbol{R}$. [2]

Theorem 2. *If $\boldsymbol{H} = (H, \cdot)$ is $\mathcal{U}_n$-injective, then the following conditions are equivalent:*

(i) $\boldsymbol{H}$ is $\mathcal{U}_n$-free.

(ii) There is a mapping $|\;| : a \mapsto |a|$ from H into the set of positive integers, such that: $b|a \Rightarrow |b| < |a|$.

(iii) Every $\mathcal{U}_n$-divisor chain in $\boldsymbol{H}$ is finite.

(iv) The set B of $\mathcal{U}_n$-primes in $\boldsymbol{H}$ is a generating set of $\boldsymbol{H}$.

Theorem 3. *The class of $\mathcal{U}_n$-free groupoids is hereditary. If k is a positive integer such that $\max\{n-1, k\} \geq 2$, then in any $\mathcal{U}_n$-free groupoid with the rang k there exist subgroupoids with infinite rang.*

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[1] $d|a$ means "d is a $\mathcal{U}_n$-divisor of a".

[2] If $n = 2$ and $|B| = 1$, then $R = B$. If $n \geq 3$ or $|B| \geq 2$, then B is a proper subset of R, and R is infinite.

Th.i will be shown in Section **i** (**i** =1,2,3), and corresponding results for the variety of commutative $\mathcal{U}_n$-groupoids will be stated in Section **4**.

Remark. It is well known that there exist $\alpha_n = (2n-2)! \,/\, n!(n-1)!$ possibilities of defining n-th powers in grupoids (see, for example: [2], III.2 or [4], I.4). Therefore, there exist α_n varieties of groupoids each of which is defined by an axiom of the form $x^n = x$. This suggests the question whether the corresponding T.1- T.3 are satisfied in the general case.

1. A CANONICAL DESCRIPTION OF FREE $\mathcal{U}_n$-GROUPOIDS

It is clear that the class $\mathcal{U}_1$ (of groupoids with $x^1 = x$) is the variety of groupoids. Therefore the notions of "$\mathcal{U}_1$-prime" and "$\mathcal{U}_1$-injectivity" have the meanings of "prime" and "injectivity" in the class of groupoids. Namely, $a \in G$ is *prime* in a groupoid $\boldsymbol{G} = (G, \cdot)$ iff $a \neq bc$ for any $b,c \in G$, and $\boldsymbol{G}$ is *injective* iff $ab = cd$ implies $a = c$, $b = d$, for any $a, b, c, d \in G$. Thus we obtain the following *characterization of free groupoids*, i.e. free $\mathcal{U}_1$-grupoids ([1], I.1).

Proposition 1.1. *A groupoid $\boldsymbol{F}=(F, \cdot)$ is free with the basis B iff $\boldsymbol{F}$ is injective, B is the set of primes in $\boldsymbol{F}$ and generates $\boldsymbol{F}$.* □

We will use the following properties of free groupoids as well.

Proposition 1.2. *If $\boldsymbol{F}$ is a free groupoid with the basis B, then the following statements hold.*

1) There exists a unique mapping $|\ |: u \mapsto |u|$, from F into the set of positive integers such that:

$$|b| = 1, \quad |uv| = |u| + |v| \tag{1.1}$$

for any $b \in B$, $u,v \in F$. (We say that $|u|$ is the *norm* of u.)

2) If $k,m \geq 1$, $u,v \in F$, then: $u^m = v^m \Leftrightarrow u = v$, and $u^k = u^m \Leftrightarrow k = m$. □

By induction on norm, the following statement can be easily shown.

Proposition 1.3. *The set R and the groupoid $\boldsymbol{R}$ are well defined by (0.3) and (0.4) - respectively; and, B is the least generating subset of $\boldsymbol{R}$. If $n = 2$ and $|B| = 1$, then $R = B$; and, if $n \geq 3$ or $|B| \geq 2$, then R is infinite and a proper subset of F.* □

Proposition 1.4. *$\boldsymbol{R} \in \mathcal{U}_n$, and B is the set of $\mathcal{U}_n$-primes in $\boldsymbol{R}$.*

Proof. Denote by u_*^k the corresponding k-th power of $u \in R$, i.e. : $u \in R \Rightarrow u_*^1 = u$, $u_*^{k+1} = u_*^k * u$. Assume that $1 \leq m < n-1$, $u_*^m = u^m$, for any $u \in R$. Then: $u_*^{m+1} = u_*^m * u = u^m u = u^{m+1}$. This implies that $u_*^{n-1} = u^{n-1}$, and thus: $u_*^n = u_*^{n-1} * u = u^{n-1} * u = u$. Thus $\boldsymbol{R} \in \mathcal{U}_n$. Clearly, an element $u \in R$ is $\mathcal{U}_n$-prime in $\boldsymbol{R}$ iff $u \in B$. □

Proposition 1.5. *$\boldsymbol{R}$ is a $\mathcal{U}_n$-free groupoid with the basis B.*

Proof. Let $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}_n$, and $\lambda: B \to G$ be an arbitrary mapping. Denote by φ the homomorphism from $\boldsymbol{F}$ into $\boldsymbol{G}$ which extends λ, and let $\psi = \varphi|R$ be the restriction of φ on R. Thus ψ extends λ, and it can be easily shown that $\psi: \boldsymbol{R} \to \boldsymbol{G}$ is a homomorphism. Thus, $\boldsymbol{R}$ is $\mathcal{U}_n$-free and B is the basis of $\boldsymbol{R}$. □

This completes the proof of Th.1.

The following statements can be also easily shown, and we will use them in the proof of Th.2.

Proposition 1.6. *$\boldsymbol{R}$ is $\mathcal{U}_n$-injective.*□

Proposition 1.7. *If u is a $\mathcal{U}_n$-divisor of v in $\boldsymbol{R}$, then $|u| < |v|$.*[3] □

Proposition 1.8. *Every $\mathcal{U}_n$-divisor chain in $\boldsymbol{R}$ is finite.*□

2. PROOF OF THEOREM 2

From Pr.1.3-1.8 we obtain the following "first half" of Th.2.

Proposition 2.1. *If $\boldsymbol{H} = (H, \cdot)$ is a $\mathcal{U}_n$-free groupoid with the basis B, then:*

1) $\boldsymbol{H}$ is $\mathcal{U}_n$-injective.

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[3] $|\ |: R \to \mathbb{N}$ is the restriction of $|\ |: F \to \mathbb{N}$ (see Pr.1.2).

2) There is a mapping $|\ |: u \mapsto |u|$ *from* H *into the set of positive integers such that:* $u\,|\,v \Rightarrow |u| < |v|$.

3) Each $\mathcal{U}_n$*-divisor chain in* $\boldsymbol{H}$ *is finite.*

4) B is the set of $\mathcal{U}_n$*-primes in* $\boldsymbol{H}$. □

Therefore:

Proposition 2.2. *If* $\boldsymbol{H}$ *satisfies the condition (i) of Th.2, then it satisfies (ii), (iii), (iv) as well.* □

The following more general statements will imply "the second part" of Th.2.

Proposition 2.3. *Let* $\boldsymbol{H} = (H, \cdot)$ *be a* $\mathcal{U}_n$*-injective groupoid. Then* $a \in H$ *is not* $\mathcal{U}_n$*-prime in* $\boldsymbol{H}$ *iff there exists a unique pair* $(b, c) \in H^2$ *such that:* $a = bc$, $b \neq c^{n-1}$. *Moreover, in this case:* $n \geq 3$ *and* $a = b^2$ *implies* $b \neq a$. □

Proposition 2.4. *If* $\boldsymbol{H} \in \mathcal{U}_n$ *and* $\boldsymbol{H}$ *satisfies the condition (ii) of Th.2, then it satisfies (iii) as well.* □

Proposition 2.5. *Assume that:* $\boldsymbol{H} \in \mathcal{U}_n$, *every* $\mathcal{U}_n$*-divisor chain in* $\boldsymbol{H}$ *is finite, and the set of* $\mathcal{U}_n$*-divisors of* a *is finite, for every* $a \in H$. *Then the following statements hold:*

a) *The set* B *of* $\mathcal{U}_n$*-primes in* $\boldsymbol{H}$ *is non-empty.*

b) *There is a mapping* $|\ |: u \mapsto |u|$ *from* H *into the set of positive integers such that* $|b| = 1$, *for each* $b \in B$, *and* $c\,|\,d \Rightarrow |c| < |d|$, *for any* $a, b \in H$.

Proof. By König Lemma (see, for example, [3],4.), the set of $\mathcal{U}_n$-divisor chains in $\boldsymbol{H}$, with a given first member is finite. Let $a \in H$, and let $|a|$ be the largest possible length of $\mathcal{U}_n$-divisor chains in $\boldsymbol{H}$ with the first member a. Then we have: $|a| = 1$ iff a is $\mathcal{U}_n$-prime in $\boldsymbol{H}$ and $c\,|\,a \Rightarrow |c| < |a|$. □

Proposition 2.6. *If* $\boldsymbol{H}$ *is a* $\mathcal{U}_n$*-injective groupoid which satisfies (iii) of Th.2, then it satisfies (iv) as well.*

Proof. By Pr.2.5, the set B of $\mathcal{U}_n$-primes in $\boldsymbol{H}$ is non-empty, and there exists a mapping $|\,|: H \to \mathbb{N}$, such that: $b \in B \Leftrightarrow |b| = 1$, and $c\,|\,a \Rightarrow |c| < |a|$ Denote by $\boldsymbol{Q}$ the subgroupoid of $\boldsymbol{H}$ generated by B. Thus $b \in H$ & $|b| = 1 \Rightarrow b \in Q$. Assume that: $c \in H$ & $|c| \leq k \Rightarrow c \in Q$, and let $a \in H$, $|a| = k+1$. Then by Pr.2.3, there exists a unique pair $b, c \in H$, such that $a = bc$, $b \neq c^{n-1}$ and $|b|, |c| \leq k$. Thus $b, c \in Q$, which implies $a = bc \in Q$. □

The following statement will complete the proof of Th.2.

Proposition 2.7. *Let* $\boldsymbol{H}$ *be a* $\mathcal{U}_n$*-injective groupoid and the set* B *of* $\mathcal{U}_n$*-primes in* $\boldsymbol{H}$ *generates* $\boldsymbol{H}$. *Then* $\boldsymbol{H}$ *is a* $\mathcal{U}_n$*-free groupoid with the basis B.*

Proof. First we have $H = \bigcup\{B_n \mid n \geq 1\}$, where $B_1 = B$ and $B_{k+1} = B_k \cup \{uv \mid u, v \in B_k\}$. By Pr.2.3, $u \in B_{k+1} \setminus B_k$ iff there exists a unique pair $(v,w) \in B_k^2$ such that: $u = vw$, $v \neq w^{n-1}$, $v, w \in B_k$ and ($v \notin B_{k-1}$ or $w \notin B_{k-1}$).

Let $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}_n$, and $\lambda: B \to G$ be a mapping. There exists a unique sequence of mappings: $(\varphi_k : B_k \to G \mid k \geq 1)$ defined in the following way: 1) $\varphi_1 = \lambda$; 2) φ_{m+1} is an extension of φ_m ; 3) If $u \in B_{k+1} \setminus B_k$ and $u = vw$, where $v \neq w^{n-1}$, then $\varphi_{k+1}(u) = \varphi_k(v)\,\varphi_k(w)$. Therefore, there exists a unique mapping $\varphi: H \to G$ such that: $u \in B_k \Rightarrow \varphi(u) = \varphi_k(u)$, and moreover $\varphi: \boldsymbol{H} \to \boldsymbol{G}$ is a homomorphism which extends λ. □

3. SUBGROUPOIDS OF $\mathcal{U}_n$-FREE GROUPOIDS

The following statements are clear.

Proposition 3.1. *If a groupoid* $\boldsymbol{H}$ *is* $\mathcal{U}_n$*-injective, then any subgroupoid of* $\boldsymbol{H}$ *is* $\mathcal{U}_n$*-injective as well.* □

Proposition 3.2. *If* $\boldsymbol{Q}$ *is a subgroupoid of a* $\mathcal{U}_n$*-groupoid* $\boldsymbol{H}$, *and if* b *is a* $\mathcal{U}_n$*-divisor of* a *in* $\boldsymbol{Q}$, *then* b *is a* $\mathcal{U}_n$*-divisor of* a *in* $\boldsymbol{H}$, *as well.* □

Proposition 3.3. *If a* $\mathcal{U}_n$*-groupoid* $\boldsymbol{H}$ *satisfies the condition (iii) of Th.2, and if* $\boldsymbol{Q}$ *is a subgroupoid of* $\boldsymbol{H}$, *then* $\boldsymbol{Q}$ *satisfies (iii), as well.* □

As a corollary, we obtain the following "first part" of Th.3.

Proposition 3.4. *If* $\boldsymbol{H}$ *is a* $\mathcal{U}_n$*-free groupoid, and* $\boldsymbol{Q}$ *is a subgroupoid of* $\boldsymbol{H}$, *then* $\boldsymbol{Q}$ *is* $\mathcal{U}_n$*-free, as well.* □

As concerns the "second part" of th.3, we will consider two cases: $n \geq 3$ and $n = 2$.

Proposition 3.5. *Let* $n \geq 3$ *and* $\boldsymbol{H}$ *be a free* $\mathcal{U}_n$*-groupoid with the one-element basis* $B = \{b\}$*, and let* $C = \{c_k \mid k \geq 1\} \subset H$ *be defined as follows:*

$$c_1 = b^2, \quad c_{k+1} = c_k b. \tag{3.1}$$

Then the rank of the subgroupoid $\boldsymbol{Q}$ *of* $\boldsymbol{H}$*, generated by* C *is infinite.*

Proof. It can be easily seen that $c_i = c_j \Rightarrow i = j$, and that C is the set of $\mathcal{U}_n$-primes in $\boldsymbol{Q}$. □

Proposition 3.6. *Let* $\boldsymbol{H} = (H, \cdot)$ *be a* $\mathcal{U}_2$*-free groupoid with two-element basis* $B = \{a,b\}$*, and let* $C = \{c_k \mid k \geq 1\} \subset H$ *be defined as follows:*

$$c_1 = ab, \quad c_{k+1} = c_k b. \tag{3.2}$$

Then C is the basis of the subgroupoid $\boldsymbol{Q}$*, generated by* C*, and C is infinite.* □

4. COMMUTATIVE $\mathcal{U}_n$-GROUPOIDS

Denote by S_n the variety of commutative $\mathcal{U}_n$-groupoids and assume that $n \geq 2$. Let $\boldsymbol{G} = (G, \cdot) \in S_n$. We say that $\boldsymbol{G}$ is S_n*-injective* iff

$$(\forall a,b)(\, a \neq b^{n-1},\ b \neq a^{n-1},\ c \neq d^{n-1},\ d \neq c^{n-1},\ ab = cd \ \Rightarrow \{a,b\} = \{c,d\}). \tag{4.1}$$

An element $a \in G$ is S_n*-prime* in $\boldsymbol{G}$ iff

$$a = bc \ \Rightarrow \ b = c^{n-1} \ \text{ or } \ c = b^{n-1}. \tag{4.2}$$

Thus $a \in G$ is *not* S_n*-prime* in $\boldsymbol{G}$ iff there exist $b,c \in G$ such that

$$a = bc \ \ \& \ \ b \neq c^{n-1} \ \ \& \ \ c \neq b^{n-1}. \tag{4.3}$$

Then b and c are called S_n*-divisors* of a in $\boldsymbol{G}$.

The results obtained above for $\mathcal{U}_n$-groupoids can be "translated" for S_n-groupoids. Here we will state only the following *canonical description of free* S_n*-groupoids.*

Theorem 1′. *Let* $\boldsymbol{F} = (F, \cdot)$ *be a free commutative groupoid with the basis B. Let R, a subset of F, and* $\boldsymbol{R} = (R, *)$ *be defined as follows:* $B \subseteq R$ *and*

$$u,v \in F \ \Rightarrow \ (uv \in R \ \Leftrightarrow \ u,v \in R \ \ \& \ \ u \neq v^{n-1} \ \ \& \ \ v \neq u^{n-1}), \tag{4.4}$$

$$u,v \in R \ \Rightarrow \ u * v = \begin{cases} uv, \text{ if } uv \in R \\ v, \text{ if } u = v^{n-1} \\ u, \text{ if } v = u^{n-1} \end{cases}$$

Then $\boldsymbol{R} = (R, *)$ *is* S_n*-free groupoid with the basis B.*□

(Th.2′ and Th.3′ can be obtained from Th.2 and Th.3 respectively, by replacing $\mathcal{U}_n$ with S_n. We will not state them explicitely.)

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СЛОБОДНИ ГРУПОИДИ СО $x^n = x$

Р е з и м е

Во работава се дава опис на класата слободни објекти во многуобразието $\mathcal{U}_n$ од групоиди коишто го задоволуваат идентитетот $x^n = x$, каде што *n* е даден природен број. Главните резултати се формулираат во §***0***, а се образложуваат во §***1***-§***3***. Соодветни резултати во врска со комутативни $\mathcal{U}_n$-групоиди се наведени во §***4***.